Math 131—Topology I

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Introduction

Math 131 is the first in a two-course undergraduate series on topology offered at Harvard University. The first two-thirds of the course thoroughly covers general point-set topology, and the remainder is spent on homotopy, monodromy, the fundamental group, and other topics.

These notes were live-TeXed, then edited for correctness and clarity. I am responsible for all errata in this document, mathematical or otherwise; any merits of the material here should be credited to the lecturer, not to me.

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The page layout for these notes is based on the layout I used back when I took notes by hand. The LaTeX styles can be found here: `https://github.com/mxw/latex-custom`.

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Lecture 2 — 9/3/10

Definition 2.1. A metric space is a set X with a map $d : X \times X \to \mathbb{R}$ such that $\forall x, y, z \in X$,

1. $d(x, y) = 0$ iff $x = y$.
2. $d(x, y) = d(y, x)$.
3. $d(x, z) \leq d(x, y) + d(y, z)$.

Example.

1. $\mathbb{R}^n$ is a metric space with the Euclidean distance $d(x, y) = \sqrt{\sum_i (x_i - y_i)^2}$.
2. $\mathbb{R}^n$ with the taxicab metric: $d(x, y) = \sum_i |x_i - y_i|$.
3. Let X be any metric space, $X_0 \subseteq X$. Then X_0 is a metric space.
4. Let X be any set. The discrete metric is given by

$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

Definition 2.2. A sequence of numbers $x_1, x_2, \ldots \in X$ a metric space with metric d is said to converge to $x \in X$ if $\forall \epsilon > 0, \exists N : d(x_n, x) < \epsilon, \forall n > N$.

Definition 2.3. Let $(X, d), (X', d')$ be metric spaces. A map $f : X \to X'$ is an isometry if $\forall x, y \in X$,

$$d'(f(x), f(y)) = d(x, y)$$

We say f is continuous if $\forall x_1, x_2, \ldots \in X$,

$$x_i \to x \Longrightarrow f(x_i) \to f(x)$$

Note. If $d'(f(x), f(y)) \leq c \cdot d(x, y)$ for some $c \geq 0$, then f is continuous.

Example. The function $\mathrm{id} : \underset{\text{taxicab}}{\mathbb{R}^n} \longrightarrow \underset{\text{Euclidean}}{\mathbb{R}^n}$ is bicontinuous but is not an isometry.

Definition 2.4. Let X be a metric space. A subset $K \subseteq X$ is closed if $\forall y_1, y_2, \ldots \in K, y_i \to x \in X \Longrightarrow x \in K$. A subset $U \subseteq X$ is open if $X - U$ is closed.

Proposition 2.5. *$U \subseteq X$ is open iff $\forall x \in U, \exists \epsilon > 0 : \forall y \in X, d(x, y) < \epsilon \Longrightarrow y \in U$.*

Proof.

$\Longrightarrow$ Suppose U open. Let $x \in U$, and try to choose points $x_1, x_2, \ldots$ such that $d(x, x_i) = \frac{1}{i}, x_i \notin U$. If we succeed, $x_i \to x$, and $x_i \in U \Longrightarrow x \in U$ by closedness. $\Rightarrow\Leftarrow$.

$\Longleftarrow$ Suppose that $\forall x \in U, \exists \epsilon > 0 : \forall y \in X, d(x, y) < \epsilon \Longrightarrow y \in U$. Choose $x_1, x_2, \ldots \in X - U$ such that $x_i \to x \in X$, and suppose $x \notin X - U$. Then $x \in U$. But then, by assumption, $x_i \not\to x$. $\Rightarrow\Leftarrow$. ∎

Theorem 2.6. *Let $(X, d), (X', d')$ be metric spaces, let $f : X \to X'$. TFAE:*

1. *f is continuous.*
2. *If $K \subseteq X'$ is closed, $f^{-1}(K)$ is closed.*
3. *If $U \subseteq X'$ is open, $f^{-1}(U)$ is open.*
4. *$\forall x \in X, \forall \epsilon > 0, \exists \delta > 0 : \forall y \in Y, d(x, y) < \delta \Longrightarrow d'(f(x), f(y)) < \epsilon$.*

Lemma 2.7. *Let X a metric space, $x \in X$. Then*

$$B_\epsilon(x) := \{y \in X : d(x, y) < \epsilon\}$$

is open.

Proof. Let $y \in B_\epsilon(x)$. This means that $d(x, y) < \epsilon$. Fix z such that $d(z, y) < \epsilon - d(x, y)$. Then $d(x, yz) \leq d(x, y) + d(y, z) < \epsilon$. Hence, by Prop., $z \in B_\epsilon(x) \Longrightarrow B_\epsilon(x)$ open. ∎

Proof. (of Theorem)

***1* $\Rightarrow$ *2*.** Say $K \subseteq X'$ is closed. Let $x_1, x_2, \ldots \in f^{-1}(K)$, $x_i \to x \in X$. Then $f(x_i) \in K, f(x_i) \to f(x)$ by continuity. $f(x) \in K \Longrightarrow x \in f^{-1}(K)$.

***2* $\Rightarrow$ *3*.** U open $\Longrightarrow X' - U$ closed $\Longrightarrow f^{-1}(X - U)$ closed $\Longrightarrow f^{-1}(U) = X - f^{-1}(X' - u)$ open.

***3* $\Rightarrow$ *4*.** Let $x \in X, \epsilon > 0$. $B_\epsilon(f(x)) \subseteq X'$ is open, and by assumption, $f^{-1}(B_\epsilon(f(x))) \subseteq X$ is open. Let $x \in f^{-1}(B_\epsilon(f(x)))$. Then $\exists \delta < 0 : B_\delta(x) \subset f^{-1}(B_\epsilon(f(x)))$. Thus, $d(x, y) < \delta \Longrightarrow d'(f(x), f(y)) < \epsilon$.

***4* $\Rightarrow$ *1*.** Suppose $x_n \to x$. Let $\epsilon < 0$. By (4), $\exists \delta > 0 : d(x, y) < \delta \Longrightarrow d'(f(x), f(y)) < \epsilon$. So, $d(x, x_i) < \delta$ for $i \gg 0$, by convergence. ∎

Lecture 3 — 9/8/10

Definition 3.1. A topological space is a set X together with a collection (a topology) $\{U\}$ of subsets of X, called open sets, such that

1. $\emptyset, X$ are open.
2. Any union of open sets is open.
3. If U and U' are open, $U \cap U'$ is open.

Definition 3.2. Alternatively, we may frame a topological space as a set X with a collection of closed subsets $\{K\}$, such that

1. $\emptyset, X$ are closed.
2. Any intersection of closed sets is closed.
3. If K and K' are closed, $K \cup K'$ is closed.

and where a subset $U \in X$ is open if $X - U$ is closed.

Example. Let (X, d) be a metric space. Then X has a topology where a subset $U \subseteq X$ is open if $X - U$ is closed.

Proof.

1. Clearly, $\emptyset$ and X are closed.
2. Say we have closed subsets $K_\alpha \subseteq X, K = \bigcap K_\alpha$. Want: K closed. Say $x_1, x_2, \ldots \in K, x_i \to x \in X$. Since $x_i \in K_\alpha, \forall i, \alpha$, and since each K_α is closed, $x \in K_\alpha \implies x \in K$.
3. Let $K, K' \subseteq X$ be closed. Say $x_1, x_2, \ldots \in K \cup K'$ converges to $x \in X$. WLOG, assume infinitely many points in the sequence $x_1, x_2, \ldots$ belong to K. $x_{i_1}, x_{i_2}, \ldots \in K$ converges to $x \implies x \in K \implies x \in K \cup K'$. ∎

Example. Let X be any set. Then X has a topology where all subsets of X are open. This is called the discrete topology and comes from the discrete metric.

Example. Let X be any set. There is a topology on X where the only open sets are $\emptyset$ and X. This is called the trivial topology.

Definition 3.3. Let X, Y be topological spaces. A map $f : X \to Y$ is continuous if $\forall U \subseteq Y$ is open, $f^{-1}(U) \subseteq X$ is open.

Example. If X has the discrete topology or Y has the trivial topology, every map $f : X \to Y$ is continuous.

Proposition 3.4. *Let X be any topological space, $X_0 \subseteq X$ any subset of X. Then*

1. *There is a smallest closed set $K \supseteq X_0$.*
2. *There is a largest open set $U \subseteq X_0$.*

Proof.

1. Let K be the intersection of all closed sets containing $X_0 \implies K$ closed.
2. Let U be the union of all open sets contained in $X_0 \implies U$ open. ∎

Definition 3.5. Define U, K as above. U is the interior of X_0, denoted $\mathring{X}_0$. K is the closure of X_0, denoted $\overline{X}_0$.

Example. Let $X = \mathbb{R}, X_0 = (0, 1]$. Then $\overline{X}_0 = [0, 1]$, $\mathring{X}_0 = (0, 1)$.

Proposition 3.6. *Let X be a metric space, $X_0 \subsetneq X$. Then $x \in \overline{X}_0$ if $\exists$ a sequence $x_n \to x$, and $x \in \mathring{X}_0$ if $\exists \epsilon > 0 : B_\epsilon(x) \in X_0$.*

Proof.

1. Let $Y = \{x \in X : \exists x_1, x_2, \ldots \in X_0, x_i \to x\} \subseteq X$. Clearly, $Y \subseteq \overline{X}_0$. Want: $Y = \overline{X}_0 \iff Y$ is closed. Let $y_1, y_2, \ldots \in Y, y_i \to y \in X$. Since $y_i \in Y, \exists x_i \in X_0 : d(x_i, y_i) < \frac{1}{2^i}$. Hence, $y_i, x_i \to y \implies y \in Y \implies Y$ closed.
2. Let $Z = \{x \in X : \exists \epsilon > 0 : B_\epsilon(x) \subseteq X_0\} \subseteq X$. We will first show that if $U \subseteq X_0$ open, then $U \subseteq Z$. If $x \notin Z$, then $B_1(x) \not\subseteq X_0, B_{1/2}(x) \not\subseteq X_0, \ldots$
$$\exists x_1 \notin X_0 : d(x, x_1) < 1$$
$$\exists x_2 \notin X_0 : d(x, x_2) < \frac{1}{2}$$
$$\vdots$$
Clearly, $x_i \to x$. But $\forall i, x_i \in X - U \implies x \in X - U$. Hence, $U \subseteq Z \implies \mathring{X}_0 \subseteq Z$. Want: $Z = \mathring{X}_0 \iff Z$ open. $\forall x \in Z, \exists \epsilon_x > 0 : B_{\epsilon_x}(x) \subseteq X_0$. $\bigcup_x B_{\epsilon_x}(x) = Z \implies Z$ open. ∎

Proposition 3.7. *Let X a topological space, $Y, Z \subseteq X$.*

1. $\overline{\emptyset} = \emptyset$
2. $Y \subseteq \overline{Y}$
3. $\overline{\overline{Y}} = \overline{Y}$
4. $\overline{Y \cup Z} = \overline{Y} \cup \overline{Z}$
5. $Y \subseteq Z \implies \overline{Y} \subseteq \overline{Z}$

Proof.

4. $\overline{Y \cup Z} \subseteq \overline{Y} \cup \overline{Z}$ by definition of closure. $Y \subseteq Y \cup Z \subseteq \overline{Y \cup Z} \implies \overline{Y} \subseteq \overline{Y \cup Z}$. Similarly, we have $\overline{Z} \subseteq \overline{Y \cup Z}$. So $\overline{Y} \cup \overline{Z} \subseteq \overline{Y \cup Z}$. ∎

Definition 3.8. A topological space is a set X together with an operator on subsets of X, $Y \to \overline{Y}$, satisfying 1, 2, 3, and 4 (5 may also be necessary).

Proof. (Sketch)
Let X be a set with such an operator. Say $Y \subseteq X$ is closed if $Y = \overline{Y}$. By axioms *1*, *2* of the new definition, axiom *1* of the original definition is satisfied. Axiom *4* of the new definition implies axiom *3* of the original. Want: $Y_\alpha = \overline{Y}_\alpha \implies \overline{\bigcap Y_\alpha} = \bigcap \overline{Y}_\alpha$. The $\supseteq$ direction is obvious. The $\subseteq$ direction is derived from new axiom 5: $\bigcap Y_\alpha \subseteq Y_\alpha \implies \overline{\bigcap Y_\alpha} \subset \overline{Y}_\alpha$. ∎

Lecture 4 — 9/10/10

Definition 4.1. Let X be a topological space, $x \in X$. A neighborhood of x is an open set $U \subseteq X$ containing x.

Observation 4.2. A subset $Y \subseteq X$ is open iff $\forall x \in Y$, there exists a neighborhood of x which is contained in Y.

Proof. $\Longrightarrow$ trivial

$\Longleftarrow$ Assume $\forall x \in Y, \exists$ a neighborhood $U_x \subseteq Y$ of x. $Y \subseteq \bigcup_{x \in Y} U_x \subseteq Y \Longrightarrow Y$ open. ∎

Definition 4.3. Let X be a topological space. A basis for the topology of X is a collection $\mathcal{B} = \{U_B \subseteq X\}$ such that

1. Each U_B is open.
2. $\forall U \subseteq X$ open, $U = \bigcup U_B$ for some $U_B \in \mathcal{B}$.

Example. Let (X, d) be a metric space. Then $\{B_\epsilon(x)\}_{\epsilon>0, x \in X}$ is a basis for the topology on X.

Proof.

1. $B_\epsilon(X)$ open. ✓
2. Let $U \subseteq X$ open. Clearly, $\bigcup_{x,\epsilon : B_\epsilon(x) \subseteq U} B_\epsilon(x) \subseteq U$. Want: $x \in U \Longrightarrow x \in B_\epsilon(y) \subseteq U$ for some $y \in X$. We can choose $y = x$ for ϵ small enough. ∎

Example. Let $X = \mathbb{R}$. The collection of sets $\{(a, b)\}_{a<b}$ is a basis for the usual topology on $\mathbb{R}$. Hence, the topology on $\mathbb{R}$ can be defined purely in terms of the ordering on $\mathbb{R}$.

Proposition 4.4. *Let X be a topological space, $Y \subseteq X$. Then Y has the structure of a topological space, where we say a subset $U \subseteq Y$ is open (in Y) if $\exists U' \in X$ open such that $U = U' \cap Y$. Equivalently, we can say $K \subseteq Y$ is closed in Y if $\exists K' \subseteq X$ closed such that $K = K' \cap Y$. This topology on Y is called the subspace topology or the induced topology.*

Proof.

1. $\emptyset = \emptyset \cap Y, Y = X \cap Y$.
2. Say $U_\alpha \subseteq Y$ open in Y. Then $U_\alpha = U'_\alpha \cap Y$ for some U'_α open in X. $V \cap Y = \bigcup_\alpha U'_\alpha \cap Y = \bigcup_\alpha U_\alpha$.
3. Say $U, V \subseteq Y$ open in Y. Then $U = U' \cap Y$, $V = V' \cap Y$ for some $U', V' \subseteq X$ open in X. $(\underbrace{U' \cap V'}_{\text{open in } X}) \cap Y = U \cap U'$. ∎

Claim 4.5. *We can always choose $K' = \overline{K}$.*

Proof. Assume $K = K' \cap Y$. $K' \subseteq X$ is closed. $K \subseteq \overline{K} \subseteq K' \Longrightarrow K \subseteq Y \cap \overline{K} \subseteq Y \cap K' = K \Longrightarrow Y \cap \overline{K} = K$. ∎

Claim 4.6. *Let X be a topological space, $Y \subseteq X$ a subspace. Let Z be any other space. A function $f : Z \to Y \subseteq X$ is continuous as a function $Z \to X$ iff it is continuous as a function $Z \to Y$.*

Proof. $f : Z \to X$ is continuous iff $\forall V \subseteq X$ open, $f^{-1}(V) \subseteq Z$ open. $f : Z \to Y$ is continuous iff $\forall V \subseteq X$ open, $f^{-1}(V \cap X) \subseteq Z$ open in Z. $f^{-1}(V) = f^{-1}(V \cap Y)$. ∎

Note. In particular, the inclusion map $Y \hookrightarrow X$ is continuous.

Proposition 4.7. *Let X, Y be topological spaces. Say a set $W \subseteq X \times Y$ is open if $\forall (x, y) \in W, \exists$ a neighborhood $U_{(x,y)} \subseteq X$ of x and a neighborhood $V_{(x,y)} \subseteq Y$ of y such that $U_{(x,y)} \times V_{(x,y)} \subseteq W$. This equips $X \times Y$ with the structure of a topological space and is called the product topology. In particular, note that*

$$W \subseteq \bigcup_{(x,y) \in W} U_{(x,y)} \times V_{(x,y)} \subseteq W$$

so $X \times Y$ has a basis of open sets of the form $U \times V, U \subseteq X$ open, $V \subseteq Y$ open.

Proof.

1. $\emptyset \subseteq X \times Y$ open, $X \times Y \subseteq X \times Y$ open.
2. Say $W = \bigcup W_\alpha$ for some W_α open. $(x, y) \in W \Longrightarrow (x, y) \in W_\alpha$. Hence, $\forall \alpha, \exists U \subseteq X$, $V \subseteq Y$ both open such that $U \times V \subseteq W_\alpha \subseteq W$.
3. Let $W, W' \subseteq X \times Y$ open. $(x, y) \in W \cap W' \Longrightarrow (x, y) \in W \Longrightarrow \exists U, V$ open such that $U \times V \subseteq W$. $(x, y) \in W \cap W' \Longrightarrow (x, y) \in W' \Longrightarrow \exists U', V'$ open such that $U' \times V' \subseteq W'$. $(\underbrace{U \cap U'}_{\text{open}}) \times (\underbrace{V \cap V'}_{\text{open}}) = (U \times V) \cap (U' \times V') \subseteq W \cap W'$. ∎

Lecture 5 — 9/13/10

Definition 5.1. Let X be a topological space, $x_1, x_2, \ldots \in X$. We say that x_n converges to a point $x \in X$ if $\forall U$ a neighborhood of x, $\exists k \in \mathbb{N} : \forall n > k$, $x_n \in U$.

Claim 5.2. *If (X, d) a metric space, metric space convergence is equivalent to topological convergence.*

Proof.

$\Longrightarrow$ Any neighborhood U of x, we can find $B_\epsilon(x) \subseteq U$ for some $\epsilon > 0$.

$\Longleftarrow$ Take $U = B_{\epsilon(x)}$. ∎

Example. Let X any set with the trivial topology. Then every sequence converges.

Definition 5.3. A topological space X is said to be Hausdorff if $\forall x, y \in X : x \neq y, \exists$ an open neighborhood U of x and an open neighborhood V of y such that $U \cap V = \emptyset$.

Example. Any set with the trivial topology is not Hausdorff.

Claim 5.4. *If X is Hausdorff, then the sequence $x_1, x_2, \ldots \in X$ converges to at most one point.*

Proof. Suppose that the claim does not hold: $x_n \to x$ and $x_n \to y$, $x \neq y$. Since X is Hausdorff, $\exists$ neighborhoods U of x and V of y, $U \cap V = \emptyset$. Then $\exists u : \forall n > u, x_n \in U$ and also $\exists v : \forall n > v, x_n \in V$. $\Rightarrow\Leftarrow$ ∎

Proposition 5.5. *Let X be any topological space. TFAE:*

1. *X is Hausdorff.*
2. *The diagonal subset, $\{(x, x) : x \in X\} \subseteq X \times X$, is closed.*

Proof.

$\Longrightarrow$ We will show that $W := \{(x, y) : x \neq y\} \subset X \times X$ is open. If $x \neq y, \exists U_{(x,y)}, V_{(x,y)}$, disjoint neighborhoods of x and y respectively. Since they are disjoint, $U_{(x,y)} \times V_{(x,y)} \subseteq W \subseteq X \times X$; by definition of the product topology, we have $U_{(x,y)} \times V_{(x,y)}$ open. $W \subseteq \bigcup_{x \neq y} U_{(x,y)} \times V_{(x,y)} \subseteq W \Longrightarrow W$ open.

$\Longleftarrow$ Assume that the diagonal subset is closed; i.e., W is open. W is a union of basic open sets of the form $U \times V$, $U, V \in X$ open. Given $(x, y) \in U \times V \subseteq W$, we have U a neighborhood of x, V a neighborhood of y. If $U \cap V \neq \emptyset, \exists z \in U \cap V \Longrightarrow (z, z) \in U \times V \subseteq W$. $\Rightarrow\Leftarrow$. ∎

Example. If (X, d) a metric space, then X is Hausdorff.

Proof. Say $x, y \in X, x \neq y$. Then $d(x, y) = r > 0$. Let $U = B_{r/2}(x), V = B_{r/2}(y)$. Suppose $U \cap V \neq \emptyset$. Then $\exists z \in U \cap V \Longrightarrow d(x, z) < \frac{r}{2}, d(z, y) < \frac{r}{2}$. But by the triangle inequality,

$$r = d(x, y) \leq d(x, z) + d(z, y) < \frac{r}{2} + \frac{r}{2} = r$$

$\Rightarrow\Leftarrow$. ∎

Proposition 5.6. *Let X be a Hausdorff space, $Y \subseteq X$. Then Y, with the induced topology, is Hausdorff.*

Proof. Let $y, y' \in Y \subseteq X : y \neq y'$. Since X is Hausdorff, $\exists U, U' \subseteq X$ neighborhoods of y, y' respectively such that $U \cap U' = \emptyset$. By definition, $U \cap Y, U' \cap Y \subseteq Y$ are open. Clearly, $(U \cap Y) \cap (V \cap Y) = \emptyset$. ∎

Proposition 5.7. *Let X, Y be Hausdorff spaces. Then $X \times Y$ with the product topology is Hausdorff.*

Proof. Let $(x, y), (x', y') \in X \times Y$ such that $(x, y) \neq (x', y')$. WLOG, assume $x \neq x'$. Since X is Hausdorff, $\exists U, U' \subseteq X$ neighborhoods of x, x' respectively sch that $U \cap U' = \emptyset$. Then $U \times Y, U' \times Y \subseteq X \times Y$ are open. We have $(x, y) \in U \times Y, (x', y') \in U' \times Y$. But $(U \times Y) \cap (V \times Y) = (U \cap V) \times Y = \emptyset \times Y = \emptyset$, as desired. ∎

Claim 5.8. *Let X be any topological space. Then $K \subseteq X$ is closed only if $x_1, x_2, \ldots \in K, x_n \to x \Longrightarrow x \in K$.*

Proof. Say $x \notin K \iff x \in X - K$ open. Then $x_n \in X - K$ for $n \gg 0$. Since $x_n \in K$, $\Rightarrow\Leftarrow$. ∎

Claim 5.9. *Let A be any linearly ordered set. Say that a subset $U \subseteq A$ is open if $\forall a \in U, \exists a' < a : (a', a] = \{b \in A : a' < b \leq a\} \subseteq U$. This forms a topology on A.*

Example. (Counterexample)

Choose A such that A has no largest element and A has no cofinal sequences (i.e., all sequences $a_1, a_2, \ldots \in A$ are bounded). Let $A^+ = A \cup \{\infty\}$, where $\forall a \in A, \infty > a$. A is not closed in A^+, but any convergent sequence in A has a limit in A.

Lecture 6 — 9/15/10

Claim 6.1. *If X is any topological space and $K \subseteq X$ is closed, then $x_1, x_2, \ldots \in K : x_n \to x \in X \Longrightarrow x \in K$. However, the converse fails in general.*

Claim 6.2. *Let X be any set. Then there is a topology on X where a set K is closed iff either $K = X$ or K is countable. We will call this the cocountable topology.*

Proof.

1. $\emptyset \subseteq X$ closed because it is countable. $X \subseteq X$ is closed.
2. Let K_α closed. If $\forall \alpha, K_\alpha = X$ then $\bigcap_\alpha K_\alpha = X$ is closed. If $\exists \beta : K_\beta \neq X$, then K_β is countable and $\bigcap_\alpha K_\alpha$ is countable, and hence closed.
3. If K, K' are closed, then $K \cup K'$ is closed. If either $K, K' = X$, then $K \cup K' = X$ is closed. If neither $K, K' = X$, then both are countable, and $K \cup K'$ is countable, and hence closed. ∎

Claim 6.3. *A sequence $x_1, x_2, \ldots \in X$, where X is taken with the cocountable topology, converges to x iff $x_n = x$ for all $n \gg 0$.*

Proof. Let $x_n \to x$, and suppose there are infinitely many indices $i_j \in \mathbb{N}$ such that $x_{i_j} \neq x$. Consider $U = X - \{x_{i_j}\}$. Since $\{x_{i_j}\}$ is countable, U is open. But $x \in U$, and hence $\nexists k : \forall n > k, x_n \in U$. $\Rightarrow\Leftarrow$. The other direction is trivial. ∎

Note. It follows that every subset $Y \subseteq X$ satisfies

(∗) If $(x_n) \in Y$ converging to $x \in X$, then $x \in Y$.

But if X is uncountable, then not every subset of X is closed. Also, note that X is not Hausdorff. Supposing $x \in U$ open, $y \in V$ open, we must have $U \cap V$ is nonempty, else $X = (X - U) \cup (X - V)$, but then X is countable.

Definition 6.4. A topological space X is first-countable if $\forall x \in X$, there is a sequence of subsets $U_1, U_2, \ldots \subseteq X$ containing x such that every neighborhood U of x contains some U_n. WLOG, assume $U_1 \supseteq U_2 \supseteq U_3 \supseteq \ldots \ni x$ (we may simply replace the sequence U_n with $U_1, U_1 \cap U_2, U_1 \cap U_2 \cap U_3, \ldots$).

Example. Let X be a metric space. The metric topology on X is first countable.

Proof. Simply consider the sequence $B_{1/n}(x)$. ■

Proposition 6.5. *Let X be a first countable topological space. Then a set $K \subseteq X$ is closed iff $x_1, x_2, \ldots \in K : x_n \to x \in X \Longrightarrow x \in K$.*

Proof.

$\Longrightarrow$ True in any topological space.

$\Longleftarrow$ Assume that $x_1, x_2, \ldots \in K : x_n \to x \in X \Longrightarrow x \in K$. Suppose $x \in X - K$. X is first countable, so $\exists U_1 \supseteq U_2 \supseteq \ldots \ni x$ such that any open neighborhood U of x contains some U_n. We claim that $U_n \subseteq X - K$ for some n. Suppose not. Then

$$\exists x_1 \in U_1, x_1 \in K$$
$$\exists x_2 \in U_2, x_2 \in K$$
$$\vdots$$

By assumption $x_n \to x$. $\forall$ neighborhoods U of x, $x_n \in U$ for $n \gg 0$. But this means $x \in K$. $\Rightarrow\Leftarrow$. So every x has some neighborhood $U_x \subseteq X - K$. Then $X - K \subseteq \bigcup_{x \in X-K} U_x \subseteq X - K$. So $X - K$ open, and K closed. ■

Proposition 6.6. *Let X, Y be topological spaces, $f : X \to Y$. If f is continuous, then f satisfies*

(∗′) $(x_n) \to x \in X \Longrightarrow f(x_n) \to f(x) \in Y$.

Moreover, X is first countable, $(*') \Longrightarrow f$ continuous.

Proof.

$\Longrightarrow$ Assume f continuous. Suppose $x_n \to x$. Want: $f(x_n) \to f(x)$. Let $U \subseteq Y$ be an open set containing $f(x)$. $f^{-1}(U) \subseteq X$ is an open set containing x. By definition of convergence, $x_n \in f^{-1}(U)$ for $n \gg 0$. Then $f(x_n) \in U \Longrightarrow f(x_n) \to x$.

$\Longleftarrow$ Assume f satisfies $(*')$, and additionally that X is first countable. Suppose $K \subseteq Y$ closed. Want: $f^{-1}(K)$ closed. By Claim 6.5 we can equivalently show that if $x_n \in f^{-1}(K)$ converges to $x \in X$, then $x \in f^{-1}(K)$. Clearly, we will have $f(x_n) \in K$, $f(x_n) \to f(x)$. Since K closed, $f(x) \in K$. So $x \in f^{-1}(K)$, which means $f^{-1}(K)$ is closed and hence f is continuous. ■

Definition 6.7. A topological space X is second-countable if there exists a countable sequence of open sets U_n which forms a basis for the topology of X.

Note. Every secound-countable space is first-countable.

Proof. Let X be a second-countable space and $\{U_n\}$ be a countable base. Let $x \in X$ and consider $\{U_i : U_i \ni x\}$. If U is a neighborhood of x, it can be written as a union of some of the U_i. Hence, $x \in U \Longrightarrow x \in U_n$ for some n, and $U_n \subseteq U$, so X is first-countable. ■

Example. Not every metric space is second-countable. For example, take X with the discrete metric. In particular, $\forall x \in X, \{x\} = B_1(x)$ is open. Hence, any basis for the topology of X must contain each $\{x\}$. If X is uncountable, the topology on X is not second-countable.

Definition 6.8. If X is a topological space, a set $Y \subseteq X$ is called dense if $\overline{Y} = X$.

Claim 6.9. *If X is second-countable, then there exists a countable dense subset of X.*

Proof. Say $U_0, U_1, U_2, \ldots$ are a basis for the topology of X, $U_0 = \emptyset$ and $U_i \neq \emptyset$ for $i \neq 0$. Then $\exists x_i \in U_i$ for $i > 0$, yielding a sequence $(x_i) \in X$. We claim that this sequence in dense. Say $U \subseteq X$ is nonempty and open. Then U is the union of some number of basis sets, which means there is some $x_i \in U$. ■

Claim 6.10. *If X is a metric space with a countable dense subset, X is second-countable.*

Proof. Consider the set $\{B_{1/n}(x_m)\}_{m,n>0}$. We claim that this is a basis for the metric topology of X. Say $U \subseteq X$ is open, let

$$U' = \bigcup_{\substack{m,n: \\ B_{1/n}(x_m) \subseteq U}} B_{1/n}(x_m) \subseteq U.$$

Want $U \subseteq U'$. Say $x \in U$. Then $\exists n : B_{1/n}(x) \subseteq U$. Then $B_{1/2n}(x) \ni x_m$ for some m. So $x \in B_{1/2n}(x_m)$. To prove that $x \in U'$, we need to know that $B_{1/2}(x_m) \subseteq U$. We know that $\forall y \in B_{1/2}(x_m), d(y, x_m) < \frac{1}{2n}, d(x, x_m) < \frac{1}{2n}$, and by the triangle inequality, $d(x, y) < \frac{1}{n}$. But this implies $B_{1/2}(x_m) \subseteq B_{1/n}(x) \subseteq U$. ■

Example. $\mathbb{R}$ is second-countable since $\mathbb{Q} \subset \mathbb{R}$ is a countable dense subset.

Lecture 7 — 9/17/10

Definition 7.1. A partially ordered set (poset) is a set A with a relation $\leq$ satisfying

1. $a \leq b, b \leq a \Longrightarrow a = b$.
2. $a \leq a$.
3. $a \leq b, b \leq c \Longrightarrow a \leq c$.

We do *not* assume that only $a \leq b$ or $b \leq a$ can hold (i.e., total ordering).

Example.

1. $\mathbb{N}$ with the ordering $0 < 1 < 2 < \dots$
2. $\mathbb{R}$
3. The collection of all subsets $S \subseteq X$ for a fixed set X, ordered by $S \leq T \iff T \subseteq S$. We will refer to this ordering as "reverse inclusion."

Remark. For a sequence (x_n), if statements P and Q have the property that

$$\exists n_0' : \forall n > n_0, P(x_n)$$
$$\exists n_0'' : \forall n > n_0, Q(x_n)$$

then we can easily conclude that $\exists n_0 : \forall n > n_0, P(x_n) \wedge Q(x_n)$. We want a notion of sequences for topological spaces that shares this behavior.

Definition 7.2. A partially ordered set A is filtered if

1. A is not empty.
2. $\forall a, b \in A, \exists c \in A : a \leq c, b \leq c$

Note that the second condition is automatic if A is totally ordered.

Example. Let X be any set. The collection of subsets of X (ordered by reverse inclusion) is filtered. (1) is clear. To show (2), note that $\forall S, T \subseteq X$, we have $S \leq S \cap T, T \leq S \cap T$. Note that $S \cap T$ is the least upper bound, but not the only upper bound (take, for example, $\emptyset$).

Claim 7.3. *Let X be a topological space, $x \in X$. Let $A := \{U \subseteq X : U \text{ open}, x \in U\}$ (ordered by reverse inclusion). Then A is filtered.*

Proof.

1. $X \in A$.
2. Let $U, V \subseteq A$. $U \cap V \in A \Longrightarrow U, V \leq U \cap V$. ∎

Remark. Note that A as defined above does not have a largest element (in particular, $\emptyset \notin A$).

Definition 7.4. Let X a topological space. A net in X is:

1. A filtered partially ordered set A
2. A function $f : A \to X$

Example. Taking $A = \mathbb{N}$ with the usual ordering, a net $f : A \to X$ yields a sequence.

Definition 7.5. A net $f : A \to X$ in X is said to converge to a point $x \in X$ if $\forall U \subseteq X$ open, $U \ni x, \exists a \in A : \forall b \geq a, f(b) \in U$.

Proposition 7.6. *Let X, Y be topological spaces, let $g : X \to Y$ be continuous. If $f : A \to X$ is a net in X converging to $x \in X$, then $g \circ f$ is a net in Y converging to $g(x)$.*

Proof. Let $U \subseteq Y$ open containing $g(x)$. We know $x \in g^{-1}(U)$ open by continuity. Since f converges, $\exists a \in A : f(b) \in g^{-1}(U), \forall b \geq a$. Equivalently, $g(f(b)) \in U$. ∎

Proposition 7.7. *Let X be a topological space, let $K \subseteq X$. Then K is closed iff K satisfies*

($*$) *If a net $f : A \to K$ converges to $x \in X$, then $x \in K$.*

Proof.

$\Longrightarrow$ Suppose K closed. Let $f : A \to K$ be a net converging to $x \in X$. Suppose for the sake of contradiction that $x \in X - K$ open. Then $\exists a \in A : \forall b \geq a$, $f(b) \in X - K$. In particular, we have $f(a) \in X - K$. But by assumption, $f(a) \in K$. $\Rightarrow\Leftarrow$. So we must have K closed.

$\Longleftarrow$ Assume ($*$) (i.e., K contains limits of all nets in K). We will show that $X - K$ open. Let $x \in X - K$, and define
$$A = \{U \subseteq X \text{ open} : U \ni x\}$$
ordered by reverse inclusion. Suppose that $\forall U \in A$, $U \cap K \neq \emptyset$. Then we can choose $x_U \in U \cap K$. This yields a net $f : A \to X$, $f(U) = x_U$. We must show that f converges; that is, given $U \ni x$ open, $\exists a \in A : \forall b \geq a$, $f(b) \in U$. Take $a = U$; then we must show $V \subseteq U \Longrightarrow x_V \in U$. But this is obvious since $x_V \in V \cap K$ and $V \subseteq U$. This means that f is actually a net in K. By ($*$), we conclude that $x \in K$. But we assumed $x \in X - K$. $\Rightarrow\Leftarrow$. So $\forall x \in X - K, \exists U_x \in A : U_x \subseteq X - K, U_x \ni x$. $X - K \subseteq \bigcup_x U_x \subseteq X - K$. This means $X - K$ open, so K closed. ∎

Corollary 7.8. *A function $g : X \to Y$ is continuous iff g sends nets converging to $x \in X$ to nets in Y converging to $g(x) \in Y$.*

Proof. $\Longrightarrow$ Done.

$\Longleftarrow$ Assume g preserves net convergence. Let $K \subseteq Y$ closed; we will show that $g^{-1}(K)$ satisfies $(*)$ and hence is also closed. By assumption, $g \circ f : A \to K$ is a net converging to $g(x)$. K closed implies $g(x) \in K \iff x \in g^{-1}(K)$. Hence $g^{-1}(K)$ closed. ■

Lecture 8 — 9/20/10

Definition 8.1. A topological space X is disconnected if $X = \emptyset$ or there exist closed subsets $K, K' \subseteq X$ such that $K \cap K' = \emptyset$, $K \cup K' = X$, and $K, K' \neq \emptyset$. X is connected if it is not disconnected. Since $K' = X - K$, we can equivalently stipulate disconnectedness if there exists a nonempty, non-X clopen set K, and similarly, X is connected if the only clopen sets in X are X and $\emptyset$ (or, rather, if there are exactly two clopen sets in X).

Theorem 8.2. *The interval* $[0,1]$ *is connected.*

Note. The set of real numbers $\mathbb{R}$ has the least upper bound property: if $S \subseteq \mathbb{R}$, $S \neq \emptyset$, S bounded above, then $\exists$ a smallest $t \in \mathbb{R} : \forall s \in S, s \leq t$.

Proof. Say $K \subseteq [0,1]$ is clopen. WLOG, assume $0 \in K$ (else take the complement). So $K \neq \emptyset$, and we want $K = X$. $K \subseteq \mathbb{R}$ is bounded above by 1, so K has a least upper bound $t \in \mathbb{R}$, $0 \leq t \leq 1$. For every $\epsilon > 0, t - \epsilon < t$, so $t - \epsilon$ is not an upper bound of K. So $\exists s \in K : t - \epsilon < s \leq t < t + \epsilon$. This means that K intersects each interval $(t-\epsilon, t+\epsilon)$ nontrivially. $\Longrightarrow t \in \overline{K}$. Since K is closed, $t \in K$. But K is also open, so for ϵ small enough, $[0,1] \cap (t-\epsilon, t+\epsilon) \subseteq K$. If $t \neq 1$, we can choose ϵ such that $t + \frac{\epsilon}{2} < 1$. $t + \frac{\epsilon}{2} \in K$. $\Rightarrow\Leftarrow$, since t is the least upper bound of K. So $t = 1 \Longrightarrow 1 \in K$.

For any $r \in [0,1]$, let $K_r = K \cap [0,r]$. So K_r is a clopen subset of $[0,1]$. $0 \in K_r \Longrightarrow K \supseteq K_r \ni r$, so $K = [0,1]$. ■

Proposition 8.3. *Let $f : X \to Y$ be a continuous surjection between topological spaces. If X is connected, then Y is connected.*

Proof. $Y \neq \emptyset$. Say $K \subseteq Y$ is clopen. Then $f^{-1}(K) \subseteq X$ is clopen. Since X is connected, $f^{-1}(K) = X$ or $f^{-1}(K) = \emptyset$. f is surjective, so $K = f(f^{-1}(K))$. So $K = f(\emptyset) = \emptyset$ or $K = f(X) = Y$. Hence, Y is connected. ■

Definition 8.4. A homeomorphism between topological spaces X, Y is a continuous map $f : X \to Y$ with a continuous inverse.

Proposition 8.5. *Let X, Y be connected topological spaces, then $X \times Y$ is connected.*

Proof. $X \times Y$ is nonempty. Say $K \subseteq X \times Y$ is clopen and nonempty. Let $(x_0, y_0) \in K$. Say $x \in X, y \in Y$; we want $(x, y) \in K$.

We will first show that $(x, y_0) \in K$. Let

$$K_{y_0} = \{(a,b) \in K, b = y_0\} \subseteq X \times \{y_0\}$$

The space $X \times \{y_0\}$ is homeomorphic to X and hence is also connected. But $K_{y_0} = K \cap (X \times \{y_0\})$, so K_{y_0} is clopen and nonempty (in the subspace topology); hence, $K_{y_0} = X \times \{y_0\}$. So $(x, y_0) \in K$.

By symmetry, we conclude similarly that $(x, y) \in K$, and hence $K = X \times Y$. ■

Definition 8.6. Let X be a topological space. A path in X is a continuous map $\gamma : [0,1] \to X$. We say γ is a path from $\gamma(0)$ to $\gamma(1)$.

Definition 8.7. A topological space X is path-connected if $X \neq \emptyset$ and $\forall x, y \in X, \exists$ a path γ from x to y.

Claim 8.8. *If X is path-connected, then X is connected.*

Proof. Suppose X is path-connected. Let $K \subseteq X$ be clopen. Suppose for the sake of contradiction that $K \neq \emptyset, K \neq X$. Then $\exists x \in K, y \notin K$. Choose a path $\gamma : [0,1] \to X, \gamma(0) = x, \gamma(1) = y$. By continuity, $\gamma^{-1}(K) \subseteq [0,1]$ is clopen. Then $0 \in \gamma^{-1}(K), 1 \notin \gamma^{-1}(K)$. $\Rightarrow\Leftarrow$, since $[0,1]$ is connected. ■

Definition 8.9. Let $X \subseteq \mathbb{R}^2$ be the set

$$X = \Big\{(x,y) \in \mathbb{R}^2 : x = 0\Big\} \cup \Big\{(x,y) : x > 0, y = \sin\frac{1}{x}\Big\}$$

X is called the topologist's sine curve.

Claim 8.10. *The topologist's sine curve (X) is connected.*

Proof. Denote the two parts of X as X_0 and X_1 respectively. $X_0 \cong \mathbb{R}$ and $X_1 \cong \{x \in \mathbb{R} : x > 0\}$. Hence, X_0 and X_1 are both path-connected. Say $K \subseteq X$ is clopen. Then $K \cap X_0$ is clopen and $K \cap X_1$ is clopen in their respective subspace topologies. So either $K = \emptyset$, $K = X_0$, $K = X_1$, or $K = X$.

However, X_0 is not open because it does not contain $B_\epsilon(0,0)$ for any $\epsilon > 0$. $B_\epsilon(0,0) \ni (\frac{1}{2\pi n}, 0)$ for every integer n, so X_1 is not closed. ■

Claim 8.11. *X is not path-connected.*

Proof. We claim that there does not exist a path from $(x, \sin\frac{1}{x}) \in X_1$ to $(0,0) \in X_0$. Suppose for the sake of contradiction that we do have such a path,

$$\gamma : [0,1] \to X : \gamma(0) = (0,0), \gamma(1) = (x, \sin\frac{1}{x})$$

Consider

$$K = \gamma^{-1}(X_0) = \{t : \gamma(t) = (0,y)\} \subseteq [0,1]$$

K is closed because X_0 is closed. Then K contains its least upper bound $t < 1$. Consider $\gamma|_{[t,1]}$. $\gamma|_{[t,1]}(t) = (0, y)$ for some y, and $\gamma|_{[t,1]}(z) \in X_1, \forall z \neq t$. Choose $\epsilon > 0$. Since γ is continuous, $\exists \delta > 0 : \forall r < \delta, d(\gamma(t+r), \gamma(t)) < \epsilon$. By the intermediate value theorem, if ϵ is small enough, our continuous path γ must have values outside the open ball of radius r. ∎

Lecture 9 — 9/23/10

Definition 9.1. Let X be a topological space. A collection of open sets $\{U_\alpha \subseteq X\}$ is a cover of X if $X = \bigcup U_\alpha$.

Definition 9.2. A topological space X is said to be compact if for every open cover $\{U_\alpha\}_{\alpha \in A}$, there exists a finite subset $A_0 \subseteq A : \{U_\alpha\}_{\alpha \in A_0}$ covers X.

Remark. Compactness can be thought of as a property which affirms that a given space does not need or cannot have any more points added to it. For instance, $\mathbb{R}^2$ is not compact, but $\mathbb{R}^2$ along with a point at infinity is compact; specifically, we have $\mathbb{R}^2 \cong S^2 - \{x\}$, where S^2 is the 2-sphere and $x \in S^2$.

Proposition 9.3. *Let X be a Hausdorff space. If $K \subseteq X$ is compact (under its subspace topology), then K is closed.*

Proof. Say $x \notin K$. Let $y \in K$; then $x \neq y$. Since X is Hausdorff, there are open neighborhoods $U_y \ni x$ and $V_y \ni y$ such that $U_y \cap V_y = \emptyset$. Note that the sets $\{V_y \cap K\}_{y \in K}$ are an open cover of K. Since K is compact, there exist finitely many points $y_1, \ldots, y_n \in K$ such that $\{V_{y_i} \cap K\}$ cover K. Let

$$U_x = \bigcap_{1 \leq i \leq n} U_{y_i}$$

U_x is open in X and contains x. $K \subseteq \bigcup V_{y_i}$, so $U_x \cap K = \emptyset$. Then $X - K \subseteq \bigcup_x U_x \subseteq X - K$. ∎

Note. We have actually proven that if $x \in X, K \subseteq X$ compact, $x \notin K$, then there are disjoint open sets U, V such that $x \in U, K \subseteq V$.

Proposition 9.4. *If $f : X \to Y$ is a surjective continuous map of topological spaces and X is compact, then Y is compact.*

Proof. Say $\{U_\alpha\}_{\alpha \in A}$ is an open cover of Y. Then $\{f^{-1}(U_\alpha)\}_{\alpha \in A}$ is an open cover of X. Since X is compact, $\exists A_0 \subseteq A$ finite such that $\{f^{-1}(U_\alpha\}_{\alpha \in A_0}$ covers X. Since f is surjective, $\{U_\alpha\}_{\alpha \in A_0}$ is an open cover of Y. ∎

Theorem 9.5. *The interval $[0, 1] \subset \mathbb{R}$ is compact.*

Proof. Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of $[0, 1]$. Let

$$S = \left\{ t \in [0, 1] : \exists A_0 \subseteq A \text{ finite s.t. } [0, t] \subseteq \bigcup_{\alpha \in A_0} U_\alpha \right\}$$

We know $0 \in S$, so S has a least upper bound $0 \leq t \leq 1$. We will show first that $t \in S$. Choose $\alpha : U_\alpha \supseteq (t - \epsilon, t + \epsilon) \cap [0, 1]$. If $t = 0$, we are done. Otherwise, we can choose ϵ such that $0 < t - \epsilon$, and since t is the least upper bound, $t - \epsilon/2 \in S$. So $[0, t - \epsilon/2] \subseteq \bigcup_{\beta \in A_0} U_\beta$, for some A_0 finite. Then

$$[0, t] \subseteq [0, t - \epsilon/2] \cup (t - \epsilon, t + \epsilon) \subseteq \bigcup_{\beta \in A_0 \cup \{\alpha\}} U_\beta$$

so $t \in S$.

If $t = 1$, we are done. Assume instead that $t < 1$. $[0, t] \subseteq \bigcup_{\alpha \in A_0} U_\alpha$. So $t \in U_\alpha$ for some $\alpha \in A_0$. For ϵ small enough, and $\frac{\epsilon}{2} < 1$, $(t - \epsilon, t + \epsilon) \cap [0, 1] \subseteq U_\alpha$. $[0, t + \epsilon/2] \subseteq [0, t] \cup (t - \epsilon, t + \epsilon) \subseteq \bigcup_{\alpha \in A_0} U_\alpha$. Then $t + \epsilon/2 \in S$. $\Rightarrow\Leftarrow$, since t is the least upper bound. So $t = 1$, and $[0, 1]$ is compact. ∎

Proposition 9.6. *If X, Y are compact topological spaces, then $X \times Y$ is compact.*

Proof. Say $\{W_\alpha\}_{\alpha \in A}$ is an open cover of $X \times Y$. For each $y \in Y$, the set $X \times \{y\}$ is compact. For each $x \in X, (x, y) \in W_\alpha$ for some α. Then there are open neighborhoods $U_{x,y} \times V_{x,y} \subseteq W_\alpha, U_{x,y} \ni x, V_{x,y} \ni y$. Fix $y \in Y$. The sets $\{U_{x,y}\}_{x \in X}$ cover X. Then there exist finitely many $x_1, \ldots, x_{k_y} \in X$ such that $X = \bigcup U_{x_i,y}$.

Let $V_y = \bigcap_y V_{x_i,y}$. So $V_y \ni y$ is open. The sets $\{V_y\}_{y \in Y}$ cover Y, so there exist $y_1, \ldots, y_n$ such that $Y = \bigcup V_{y_j}$. So

$$X \times Y \subseteq \bigcup_{1 \leq j \leq n} \bigcup_{1 \leq i \leq k_{y_j}} U_{x_i,y_j} \times V_{x_i,y_j}$$

Each of these product neighborhoods was constructed to lie in W_α for some α. So $X \times Y$ is contained in a union of finitely many of the U_α. ∎

Example. Let $K \subseteq \mathbb{R}^n$. Then K is compact iff K is closed and bounded.

Proof.

$\Longrightarrow$ If K is compact, K is closed since $\mathbb{R}^n$ is Hausdorff. $K \subseteq \bigcup_{n>0} B_n(0)$. If K is compact, we only need finitely many. So $K \subseteq B_n(0) \Longrightarrow K$ is bounded.

$\Longleftarrow$ Say K is closed and bounded. Then $K \subseteq [-A, A]^n \subseteq \mathbb{R}^n$ for $A \gg 0$. $[-A, A]$ is compact since $[0, 1]$ is compact and $[-A, A] \cong [0, 1]$. And $[-A, A]^n$ is compact by the previous result. Compactness follows from the subsequent lemma. ∎

Lemma 9.7. *If X is a compact topological space, $K \subseteq X$ is closed, then K is compact.*

Proof. $K \subseteq \bigcup_{\alpha\in A} U_\alpha$. $X = K \cup (X - K) \subseteq (\bigcup_\alpha U_\alpha) \cup (X - K)$. X is compact, so $X \subseteq \bigcup_{\alpha\in A_0} U_\alpha \cup (X - K)$. Since $X - K$ open, $K \subseteq \bigcup_{\alpha\in A_0} U_\alpha$ (for a slightly different A_0). ■

Lecture 10 — 9/24/10

Definition 10.1. Let X be a topological space and let $x_1, x_2, \ldots \in X$. An accumulation point of $x_1, x_2, \ldots$ is a point $y \in X$ such that every open neighborhood $U \ni y$ contains infinitely many of the x_i's.

Proposition 10.2. *Suppose that X is first-countable. Then the sequence $x_1, x_2, \ldots \in X$ has $y \in X$ as an accumulation point iff it has a subsequence $x_{i_1}, x_{i_2}, \ldots$ converging to y.*

Proof.

$\Longleftarrow$ Assume there is a subsequence $x_{i_1}, x_{i_2}, \ldots$ converging to $y \in X$. Then any open neighborhood $U \ni y$ contains infinitely many terms x_{i_k}, and hence infinitely many x_i. Note that this direction does not require the assumption of first-countability.

$\Longrightarrow$ Assume y is an accumulation point. X is first countable, so there is a sequence of open sets $U_1 \supseteq U_2 \supseteq \ldots \ni y$ such that any open neighborhood $U \ni y$ contains some U_i. Then U_1 contains infinitely many x_i, and hence contains some point x_{i_1}. Similarly, U_2 contains x_{i_2}, and we can choose x_{i_2} such that $i_2 > i_1$. And $U_3 \ni x_{i_3}, i_3 > i_2$. Continuing in this manner yields a subsequence $x_{i_1}, x_{i_2}, x_{i_3}, \ldots$.

Take any open neighborhood $V \ni y$. By our choice U_i (by first-countability), $\exists n \in \mathbb{N} : U_n \subseteq V$. Then $\forall m \geq n, x_{i_m} \in U_m \subseteq V$. So $\forall m \geq n, x_{i_m} \in V$. Thus, $x_{i_m} \to y$. ■

Definition 10.3. Recall that a topological space X is compact if every open cover $\{U_\alpha\}_{\alpha\in A}$ has a finite subcover. Equivalently, X is compact if, for any collection of closed sets $\{K_\alpha\}_{\alpha\in A}$, if every finite subset of the K_α have a common point, then all the K_α have a common point. That is,

$$\bigcap_{\alpha\in A_0} K_\alpha \neq \emptyset \forall A_0 \subseteq A \text{ finite} \Longrightarrow \bigcap_{\alpha\in A} K_\alpha \neq \emptyset$$

Proposition 10.4. *Let X be a compact topological space. Then any sequence of points in X has an accumulation point (and if X is first-countable, then every sequence has a convergent subsequence.)*

Proof. For each $n \geq 0$, let K_n be the closure of the set $\{x_n, x_{n+1}, \ldots\}$. Note that if $A_0 \subseteq \{1, 2, \ldots\}$ is finite, then A_0 has a largest element n.

$$\bigcap_{m\in A_0} K_m = K_n \neq \emptyset$$

If X is compact, then using the above definition,

$$\bigcap_{n>0} K_n \neq \emptyset$$

Take $y \in \bigcap_n K_n$. Then $\forall n, y \in K_n$. So for every n and every open $U \ni y$,

$$\emptyset \neq U \cap \{x_n, x_{n+1}, \ldots\}$$

Since this is true of every n, U contains infinitely many x_i. ■

Theorem 10.5. *Let (X, d) be a metric space. TFAE:*

1. *X is compact.*
2. *Every sequence in X has an accumulation point.*
3. *Every sequence in X has a convergent subsequence.*
4. *(a) $\forall\epsilon > 0$, there is a finite cover of X by B_ϵ's.*
   (b) $\forall\{U_\alpha\}$ an open cover of X, there exists $\epsilon > 0$ such that every B_ϵ is contained in some U_α.

Proof.

1 ⇒ 2. ✓

2 ⇒ 3. ✓(X is first-countable)

3 ⇒ 4. Assume *3*. We will show (a). Choose $\epsilon > 0$. Pick $x_1 \in X$. Choose $x_2 \in X$ such that $d(x_2, x_1) \geq \epsilon$ if possible. Choose $x_3 \in X$ such that $d(x_3, x_1), d(x_3, x_2) \geq \epsilon$ if possible. But condition *3* implies that this process must stop at some $n \in \mathbb{N}$. So X is covered by those n ϵ-balls.

Now we will show (b). Say $\{U_\alpha\}$ covers X. Suppose that $\forall n, \exists B_{1/n}(x_n)$ for some x_n such that $\forall\alpha, B_{1/n}(x_n) \not\subseteq U_\alpha$. We will show this sequence $x_1, x_2, \ldots$ does not have a convergent subsequence. Suppose otherwise. Say $x_{i_k} \to x \in X$. $x \in U_\alpha$ for some α. And $B_\epsilon(x) \subseteq U_\alpha$ for some ϵ. We know that $d(x, x_{i_k}) < \frac{\epsilon}{2}$ for $k \gg 0$. By the triangle inequality, $B_{\epsilon/2}(x_{i_k}) \subseteq U_\alpha$. But by construction, we must have $\frac{\epsilon}{2} < \frac{1}{i_k}$. $\Rightarrow\Leftarrow$, for $k \gg 0$.

4 ⇒ 1. Assume $(a), (b)$. Let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of X. By (b), $\exists\epsilon > 0 : \exists\alpha \in A$, $B_\epsilon(x) \subseteq U_\alpha$. By (a), there exists a finite cover of X by $B_\epsilon(x_1), B_\epsilon(x_2), \ldots, B_\epsilon(x_n)$. But each

$B_\epsilon(x_i) \subseteq U_{\alpha_i}$. If we take $A_0 = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, we know that

$$\bigcup_{\alpha \in A_0} \supseteq \bigcup_{i<n} B_\epsilon(x_i) \supseteq X \qquad \blacksquare$$

Definition 10.6. Let X be a topological space, $f : A \to X$ a net. An accumulation point of f is a point $y \in X$ such that $\forall U \ni y$ an open neighborhood, $\exists a \in A, \forall b \geq a : f(b) \in U$.

Theorem 10.7. *Let X be a topological space. TFAE:*

1. *X is compact.*
2. *Every net in X has an accumulation point.*

Proof. $\Longleftarrow$ Left as exercise.

$\Longrightarrow$ Say $f : A \to X$ is a net. For every $\alpha \in A$, let K_a be the closure of the set $\{f(b) : b \geq a\}$. Recall that $\forall \alpha_1, \ldots, \alpha_n \in A, \exists \alpha \geq \alpha_i$. So

$$f(\alpha) \in K_\alpha \subseteq \bigcap K_{\alpha_i}$$

Then $\bigcap K_{\alpha_i}$ is nonempty for any finite set of α_i. Since X is compact,

$$\bigcap_{\alpha \in A} K_\alpha \neq \emptyset$$

But this is precisely the set of accumulation points of X. $\blacksquare$

Lecture 11 — 9/27/10

Definition 11.1. Let $\{X_\alpha\}_{\alpha \in A}$ be a collection of topological spaces. We know that the product of sets

$$X = \prod_{\alpha \in A} X_\alpha = \{(x_\alpha \in X_\alpha)_{\alpha \in A}\}$$

We can define a topology on X having a basis of open sets

$$\prod_{\alpha \in A} U_\alpha$$

where each $U_\alpha \overset{\circ}{\subseteq} X_\alpha$. This is called the box topology.

Observation 11.2. Suppose each $X_\alpha = Y$. Then

$$X = \prod_\alpha X_\alpha = \{f : A \to Y\} = Y^A$$

There is a canonical map

$$\begin{aligned} \Delta : Y &\longrightarrow Y^A \\ y &\longmapsto f : \forall a \in A, f(a) = y \\ y &\longmapsto (y \in X_\alpha)_{\alpha \in A} \end{aligned}$$

But $\Delta : Y \to Y^A$ is usually not continuous if Y^A has the box topology. We know that

$$\bigcap_{\alpha \in A} U_\alpha = \Delta^{-1}\left(\prod_\alpha U_\alpha\right)$$

Take $A = \mathbb{N}, Y = [0,1], Y^A = \{(t_0, t_1, \ldots) : 0 \leq t_i \leq 1\}$. We can have an open set (for the box topology) consisting of sequences with $t_i < \frac{1}{i}$

$$U = [0,1] \times [0,1) \times [0, \frac{1}{2}) \times [0, \frac{1}{3})$$

$(0,0,0,\ldots) \in U$, but $\forall \epsilon > 0, (\epsilon, \epsilon, \epsilon, \ldots) \notin U$.

Definition 11.3. Let $\{X_\alpha\}_{\alpha \in A}$ be a collection of topological spaces, $X = \prod_\alpha X_\alpha$. The product topology on X has as a basis of open sets all sets of the form

$$\prod_{\alpha \in A} U_\alpha$$

where each $U_\alpha \subseteq X_\alpha$ open and $U_\alpha = X_\alpha$ for all but finitely many $\alpha \in A$.

Claim 11.4. *This defines a topology on X, where a set $U \subseteq X$ is open if $\forall (x_\alpha) \in U, \exists U_\alpha \subseteq X_\alpha$ open such that $U_\alpha = X_\alpha$ for all but finitely many $\alpha \in A$ and $x_\alpha \in U_\alpha$ for all α, and such that $\prod U_\alpha \subseteq U$.*

Proof.

1. $\emptyset, X$ are clearly open.
2. Given any collection of sets U_i, each of which contain a product of sets as given above, their union clearly contains such a product of sets.
3. Let U, V be open, $(x_\alpha) \in U \cap V$. $\exists U_\alpha \subseteq X_\alpha$ open, $x_\alpha \in U_\alpha, U_\alpha = X_\alpha$ for almost all α, such that $\prod U_\alpha \subseteq U$. $\exists V_\alpha \subseteq X_\alpha$ open, $x_\alpha \in V_\alpha, V_\alpha = X_\alpha$ for almost all α, such that $\prod V_\alpha \subseteq V$.

$$\prod U_\alpha \cap V_\alpha = \prod U_\alpha \cap \prod V_\alpha \subseteq U \cap V$$

$U_\alpha \cap V_\alpha = X_\alpha$ for almost all α. $\blacksquare$

Claim 11.5. *Let Y be a topological space, A a set. The map*

$$\Delta : Y \to Y^A = \{\prod_{\alpha \in A} Y\}$$

is continuous if Y^A is given the product topology.

Proof. Want: $\Delta^{-1}U \subseteq Y$ is open if U is a basic open set for the product topology. Choose such a U; then $U = \prod_\alpha U_\alpha$. We know that $\Delta^{-1}U = \bigcap_{\alpha \in A} U_\alpha$ by definition of Δ. By assumption, $U_\alpha = Y$ for all α outside a finite set $A_0 \subseteq A$.

$$\bigcap_{\alpha \in A} U_\alpha = \bigcap_{\alpha \in A_0} U_\alpha$$

is open. $\blacksquare$

Proposition 11.6. *Recall that a map $f : X \to Y \times Z$ is continuous iff the maps $f_Y : X \to Y, f_Z : X \to Z$ are continuous ($f = (f_Y, f_Z)$). Now let $\{X_\alpha\}_{\alpha\in A}$ be a collection of topological spaces, let Y be any topological space. Let $f : Y \to X =: \prod_\alpha X_\alpha$. We can write $f = (f_\alpha : Y \to X_\alpha)$. If X is given the product topology, then f is continuous iff each f_α is continuous.*

Proof.

$\Longrightarrow$ Say f is continuous. Want: $f_\alpha^{-1}(U_\alpha) \subseteq Y$ is open for each open $U_\alpha \subseteq X_\alpha$. Let $U = U_\alpha \times \prod_{\beta\neq\alpha} X_\beta$; it is open.

$$f_\alpha^{-1}(U_\alpha) = f^{-1}(U)$$

is open by continuity.

$\Longleftarrow$ Assume each f_α is continuous. Want: $f^{-1}(U) \subseteq Y$ is open if $U \subseteq X$ is open. Assume that U is a basic open set, $U = \prod U_\alpha$, $U_\alpha \overset{\circ}{\subseteq} X_\alpha$, and $U_\alpha = X_\alpha$ for all $\alpha \notin A_0 \subseteq A$ a finite subset.

$$f^{-1}(U) = \bigcap_{\alpha\in A} f_\alpha^{-1}(U_\alpha) = \bigcap_{\alpha\in A_0} f_\alpha^{-1}(U_\alpha)$$

is open. ■

Example. There exists a topological space X and a subset $Y \subseteq X$ such that $(*)$ every sequence $x_1, x_2, \ldots \in Y$ converging to $x \in X$ implies that $x \in Y$, but Y is not closed.

Proof. Let

$$X = \prod_{\alpha\in A} \{0, 1\} = \{0, 1\}^A$$

where $\{0, 1\}$ has the discrete topology and X has the product topology. $x_1, x_2, \ldots \in X$ converges to $x \in X$ iff for every $\alpha \in A$, the sequence $x_1(\alpha), x_2(\alpha), \ldots \subseteq \{0, 1\}$ converges to $x(\alpha)$. Let

$$Y = \{x \in X : x(\alpha) = 1 \text{ for only countably many } \alpha\}$$

Y satisfies $(*)$. But Y is not closed if A is uncountable. In fact, Y is dense in X, so for any nonempty $U \overset{\circ}{\subseteq} X, Y \cap U \neq \emptyset$. ($Y$ would be dense even if the condition imposed read "finitely many" rather than "countably many," since any open set of X has the requirement that for every element of X, all but finitely many components can be either 0 or 1.) But $Y \neq X$ because $(x_\alpha = 1)_{\alpha\in A}$ is not in Y, and since Y is dense it is not closed. ■

Proposition 11.7. *Let X be a product of topological spaces $\{X_\alpha\}_{\alpha\in A}$. If each X_α is Hausdorff, then X is Hausdorff.*

Proof. Let $x = (x_\alpha), y = (y_\alpha) \subseteq X : x \neq y$. Then $\exists \alpha \in A : x_\alpha \neq y_\alpha$. X_α is Hausdorff. Then $\exists U \ni x_\alpha, V \ni y_\alpha$ open in X_α such that $U \cap V = \emptyset$. Consider $U \times \prod_{\beta\neq\alpha} U_\beta, V \times \prod_{\beta\neq\alpha} X_\beta$, which are open sets in X. Their intersection is empty because $U \cap V = \emptyset$. ■

Theorem 11.8 (Tychonoff's Theorem)**.** *Let X be a product of topological spaces X_α. If each X_α is compact, then X is compact.*

Example. The space $\{0, 1\}^A$ is compact for any set A.

Example. The space $[0, 1]^A$ is compact and Hausdorff for any set A.

Theorem 11.9. *Any compact Hausdorff space X is homeomorphic to a closed subset of $[0, 1]^A$ for some set A.*

Lecture 12 — 9/29/10

Lemma 12.1 (Zorn's Lemma)**.** *Let A be a partially ordered set such that every linearly ordered subset of A has an upper bound. Then A has a maximal element. That is, $\exists a \in A : a \leq b \iff b = a$.*

Definition 12.2. A linearly ordered set A is well-ordered if every nonempty subset of A has a least element.

Example.

1. Any finite linearly ordered set has a least element.
2. $\mathbb{N}$ is well-ordered.
3. $\mathbb{Z}$ is not well-ordered.

Note. If A is well-ordered, then $A \cup \{\infty\}$, where we consider $\infty > A$, is well-ordered.

Example. The set $\{0 < 1 < 2 < \cdots < \omega\}$ is well-ordered. Continuing to add maximal elements, we end up with a set

$$\{0 < 1 < 2 < \cdots < \omega < \omega + 1 < \omega + 2 < \ldots\}$$

that is well-ordered.

Claim 12.3. *Let A be a linearly ordered set. TFAE:*

1. *A is well-ordered.*
2. *There does not exist an infinite descending sequence $a_0 > a_1 > a_2 > \ldots \in A$.*

Proof.

$\Longrightarrow$ Assume that A is well-ordered. The set $\{a_0 > a_1 > \ldots\}$ clearly has no smallest element. $\Rightarrow\Leftarrow$.

$\Longleftarrow$ Assume that there is no infinite descending sequence $a_0 > a_1 > a_2 > \ldots \in A$. Consider some nonempty subset $S \subseteq A$. Choose $a_0 \in S$. If a_0 is minimal in S, then we're done; otherwise, choose $a_1 < a_0$. Continuing in this way, we either stop and find a smallest element of S, or we do not stop and come to an infinite descending sequence. $\Rightarrow\Leftarrow$. ∎

Proposition 12.4 (Principle of Transfinite Recursion)**.** *Let A be any well-ordered set, and suppose we want to define $f : A \to K$ for some set K. Suppose we are given a "rule" for computing*

$$f(a) = \text{something depending on } \{f(b)\}_{b<a}$$

Then there exists a unique function $f : A \to K$ satisfying this rule.

Proof. Let

$$S = \{a \in A \mid \exists! f_a : \{b \in A : b \le a\} \to K, \text{ satisfying rule}\}$$

We claim that $S = A$. Suppose otherwise; then $A - S$ has a smallest element a. $\forall b < a, f_b : \{c \in A : c \le b\} \to K$ is well-defined. By uniqueness, if $b \le b'$, then

$$f_b = f_{b'}|_{\{c\in A : c\le b\}}$$

Then

$$f_{<a} := \bigcup_{b<a} f_b : \{c \in A : c < a\} \to K$$

Note that $f_{<a}$ extends uniquely to a function

$$f_a : \{c \in A : c \le a\} \to K$$

satisfying our recursion. Then $a \in S$. $\Rightarrow\Leftarrow$. So $S = A$. By the argument above, we get a function

$$\bigcup_{b\in A} f_b : A \to K$$ ∎

Example. Let A and B be well-ordered sets. Try to define a map $f : A \to B$ by $f(a) = \min\{b \in B : b \neq f(a'), a' < a\}B$. There are two cases:

1. f is well-defined, and we get an injective map $f : A \to B$ whose image is downward-closed in B.

2. f is not well-defined. Then $\exists a \in A$ such that $f : \{a' \in A : a' < a\} \to B$ is an isomorphism, and we have a map $f^{-1} : B \to \{a' \in A : a' < a\} \subseteq A$ that is downward-closed by construction.

We have shown that given any two well-ordered sets A, B, one is uniquely isomorphic to a downward-closed subset of the other.

Definition 12.5. We say that two well-ordered sets are equivalent if they are isomorphic. The equivalence classes of this relation are called <u>ordinals</u>. (In general, the equivalence classes of linearly ordered sets are called <u>order types</u>.)

Note. The ordinals are linearly ordered by saying $A \le B$ if there exists an isomorphism from A to a downward-closed subset of B.

Observation 12.6. If A is well-ordered with order type α, then $A \cong \{\beta : \beta < \alpha\}$, the ordered set of ordinals less than α. Note that $\beta < \alpha$ iff β is the order type of a downward-closed subset $B \subsetneq A$. $A - B$ has a least element a. Then $B = \{b \in A : b < a\}$. That is, the map

$$A \ni a \mapsto \{b \in A : b < a\}$$

is an isomorphism $A \to \{\beta : \beta < \alpha\}$.

Claim 12.7. *The collection of all ordinals is well-ordered.*

Proof. Say $\alpha_0 > \alpha_1 > \alpha_2 > \cdots$. α_0 is the order type of some well-ordered set A. $A \cong \{\beta : \beta < \alpha_0\}$. But this would be an infinite decreasing sequence in a well-ordered set A. So the ordinals must be well-ordered. ∎

Claim 12.8. *Let S be a set. Let $\mathrm{Ord}(S) = \{\alpha : \alpha$ is the order type of an ordering on some subset of $S\}$. Then $|\mathrm{Ord}(S)| > |S|$; that is, there exists no injection $\lambda : \mathrm{Ord}(S) \to S$.*

Example. Say $S = \{x_1, x_2, \ldots, x_n\}$. Then $\mathrm{Ord}(S) =$ order types of $\emptyset, \{1\}, \{1, 2\}, \ldots, \{1, \ldots, n\}$.

Proof. Say λ exists as above. Then $\lambda(\mathrm{Ord}(S)) \subseteq S$ is linearly ordered via the ordering on $\mathrm{Ord}(S)$. Then the order type of $\mathrm{Ord}(S) \in \mathrm{Ord}(S)$ (is an element). But this order type is also the least ordinal not in $\mathrm{Ord}(S)$. $\Rightarrow\Leftarrow$. ∎

Lemma 12.9 (Zorn's Lemma)**.** *Let S be a partially ordered set such that any linearly ordered subset of S has an upper bound. Then S has a maximal element.*

Proof. Suppose for the sake of contradiction that S does not have a maximal element. We will define an injective map $\lambda : \mathrm{Ord}(S) \to S$ by transfinite recursion. By our assumption, for any $S' \subseteq S$ linearly ordered, we can choose an upper bound $x_{S'} \notin S'$ of S'. Define $\lambda(\alpha) = x_{S'}$, where $S' = \{\lambda(\beta) : \beta < \alpha\}$. By transfinite recursion, $\lambda(\alpha)$ is defined for all α. But λ cannot exist because $\mathrm{Ord}(S)$ is too big. $\Rightarrow\Leftarrow$. Hence, S has a maximal element. ∎

Lecture 13 — 10/1/10

Theorem 13.1 (Tychonoff's Theorem)**.** *Any product of compact topological spaces is compact.*

Lemma 13.2 (Zorn's Lemma)**.** *If A is a partially ordered set such that every linearly ordered subset of A is bounded above, then A has a maximal element.*

Remark. Recall that a topological space X is compact if any collection $\{K_\alpha\}_{\alpha\in S}$ of closed sets satisfying the so-called finite intersection property, by which we mean

$$\forall S_0 \subseteq S \text{ finite}, \bigcap_{\alpha\in S_0} K_\alpha \neq \emptyset,$$

implies that $\bigcap_{\alpha\in S} K_\alpha \neq \emptyset$. More generally,

$(*)$ $\forall \{K_\alpha\}_{\alpha\in S} : \bigcap_{\alpha\in S_0} K_\alpha \neq \emptyset \Longrightarrow \bigcap_{\alpha\in S} \overline{K}_\alpha \neq \emptyset$

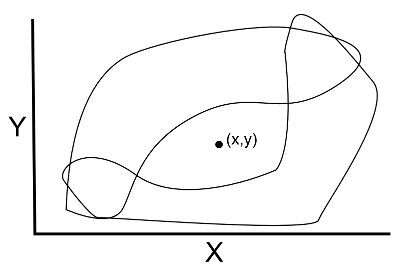
Note that this condition is both necessary and sufficient for compactness.

Example. We will explore a direction for the proof in the case of $X \times Y$. Say X, Y are compact, say $\{K_\alpha \subseteq X \times Y\}$ satisfying $(*)$. Want: $\bigcap \overline{K}_\alpha \neq \emptyset$.

Look at $\{\pi_X(K_\alpha) \subseteq X\}$. Note that

$$\bigcap_{\alpha\in S_0} \pi_X(K_\alpha) \supseteq \pi_X\left(\bigcap_{\alpha\in S_0} K_\alpha\right) \neq \emptyset$$

So $\exists x \in \bigcap_{\alpha\in S_0} \pi_X(K_\alpha)$, and hence these sets satisfy $(*)$. But simply choosing such an x and y fails to provide us necessarily with a suitable point (x, y).



Proof. To arrive at our desired conclusion, we will try to strengthen the condition imposed on our choice of x, y. Let us enlarge the set $\{K_\alpha\}_{\alpha\in S}$, which we will call $\mathcal{S}$. Let $\mathscr{A}$ be the collection of all sets $\mathcal{S}' \subseteq \mathcal{P}(X)$ with the finite intersection property.

Let $X = \prod_{i\in I} X_i$ where each X_i is compact. Want: $\mathcal{S} \in \mathscr{A} \Longrightarrow \bigcap_{\alpha\in\mathcal{S}} \overline{K}_\alpha \neq \emptyset$. Given such a collection $\mathcal{S}$, we claim the following lemma:

Lemma 13.3. *$\mathcal{S}$ is contained in a larger $\mathcal{S}_{\max} \in \mathscr{A}$, $\mathcal{S}_{\max} \subseteq \mathcal{P}(X)$ such that $\mathcal{S}_{\max}$ is maximal.*

Let $\mathscr{A}' = \{\mathcal{S}' \in \mathscr{A} : \mathcal{S} \subseteq \mathcal{S}'\}$. We will show that every linearly ordered subset of $\mathscr{A}'$ has an upper bound (with the partial ordering of *forward* inclusion). Say $\{\mathcal{S}_\beta\}_{\beta\in B}$ is a linearly ordered collection of elements of $\mathscr{A}'$. Let $\mathcal{S}' = \bigcup \mathcal{S}_\beta$. We want to have $\mathcal{S}' \in \mathscr{A}'$; that is, we want to know that if $\mathcal{S}_0 \subseteq \mathcal{S}'$ is finite, then $\bigcap_{K\in\mathcal{S}_0} K \neq \emptyset$. But $\mathcal{S}_0 \subseteq \mathcal{S}_\beta$ for β large, and since $\mathcal{S}_\beta \in \mathscr{A}'$, our conclusion follows. By Zorn's lemma, there exists a maximal $\mathcal{S}_{\max} \in \mathscr{A}$.

Now we can equivalently show that if $\mathcal{S} \in \mathscr{A}$ is maximal, then $\bigcap_{K\in\mathcal{S}} \overline{K} \neq \emptyset$. Let us now show that

Lemma 13.4. *$\mathcal{S}$ is closed under finite intersection, and given a subset $Y \subseteq X$, if $\forall K \in \mathcal{S}, Y \cap K \neq \emptyset$, then $Y \in \mathcal{S}$.*

If $Y, Y' \in \mathcal{S}$, either $\mathcal{S} \cup \{Y \cap Y'\} = \mathcal{S}$ and we are done, or $\mathcal{S} \cup \{Y \cap Y'\} \supsetneq \mathcal{S}$. Suppose the latter. Since $\mathcal{S}$ is maximal, $\mathcal{S} \cup \{Y \cap Y'\} \notin \mathscr{A}$. Then $Y \cap Y' \cap \bigcap_{K\in\mathcal{S}_0} K = \emptyset$. $\Rightarrow\Leftarrow$. Now we claim that $Y \in \mathcal{S}$ as given above. Otherwise, $\mathcal{S} \cup \{Y\} \supsetneq \mathcal{S}$. Then, taking finitely many elements $K_j \in \mathcal{S} \cup \{Y\}$, either $K_j \neq Y$, in which case $\bigcap K_j \neq \emptyset$, or $\exists r : K_r = Y$, which means that $\bigcap K_j = Y \cap \bigcap_{j\neq r} K_j \neq \emptyset$ because $\mathcal{S}$ is closed under intersection.

For $i \in I$, let $\pi_i : X \to X_i$ be the projection of X onto X_i. The collection of sets $\{\pi_i(K) : K \subseteq \mathcal{S}\}$ satisfies the finite intersection condition:

$$\bigcap_{K\in\mathcal{S}_0} \pi_i(K) \supseteq \pi_i\left(\bigcap_{K\in\mathcal{S}_0} K\right) \neq \emptyset$$

Since X_i is compact, we can choose

$$x_i \in \bigcap_{K\in\mathcal{S}} \overline{\pi_i(K)}$$

Let $x \in X$ be such that $\pi_i(x) = x_i$. Finally, we want that $x \in \bigcap_{K\in\mathcal{S}} \overline{K}$. Equivalently, we want for every $U \ni x$ an open neighborhood, then $\forall K \in \mathcal{S}, U \cap K \neq \emptyset$. WLOG, assume that U is basic; that is, $U = \prod U_i$ where each $U_i \subseteq X_i$ is an open neighborhood $U_i \ni x_i$, and $U_i = X_i$ for $i \notin I_0$ finite.

By construction, $x_i \in \bigcap \overline{\pi_i(K)}$. So $U_i \cap \pi_i(K)$ nontrivially for all $K \subseteq \mathcal{S}$ at a point $\pi_i(y)$. Then $\forall K \subseteq \mathcal{S}$, $\exists y_K \in \pi_i^{-1}(U_i) \cap K$. Then $\pi_i^{-1}(U_i) \in \mathcal{S}$, by Lemma 13.4. We know that $U = \bigcap_{i\in I} \pi_i^{-1}(U_i)$. But $\forall i \notin I_0$ finite, $\pi_i^{-1}(U_i) = X$. Hence, $U = \bigcap_{i\in I_0} \pi_i^{-1}(U_i)$. Then $U \in \mathcal{S}$, by Lemma 13.4. So $\forall K \subseteq S, U \cap K \neq \emptyset$. Then $x \in \overline{K}$, and hence

$$x \in \bigcap_{K\subseteq\mathcal{S}} \overline{K}$$

and hence, X is compact. ■

Definition 13.5. Let X be a topological space. A collection of open sets $\{U_i\}$ is a subbasis for X if every open set $U \subseteq X$ has the form $\bigcup U_\alpha$ where each U_α is a finite intersection of the U_i.

Example. If $X = \prod X_\alpha$, then the sets of the form $\{\pi_\alpha^{-1}(U_\alpha)\}_{U_\alpha \subseteq X_\alpha \text{ open}}$ are a subbasis for X.

Lemma 13.6 (Alexander Subbase Theorem)**.** *Let X be a topological space. If X has a subbasis $\{U_i\}$ such that every cover of X by sets of the form U_i has a finite subcover, then X is compact.*

Proof. (Alternate Proof of Tychonoff's Theorem)
Let $X = \prod X_\alpha$, where each X_α is compact. Say that X is covered by sets of the form $\pi_\alpha^{-1}(U_{\alpha,i})$. Suppose this has no finite subcover. Then $\forall\alpha$, the $U_{\alpha,i}$ can have no finite subcover. Since X_α is compact, $\exists x_\alpha \in X_\alpha : x \notin U_{\alpha,i}$. We can choose $x \in X : \pi_\alpha(x) = x_\alpha$. Then $x \notin \pi_\alpha^{-1}(U_{\alpha,i})$. $\Rightarrow\Leftarrow$. ∎

Lecture 14 — 10/4/10

Theorem 14.1 (Alexander Subbase Theorem)**.** *If X is a topological space having a subbasis $\{U_i\}_{i\in I}$ and every cover of X by open sets in $\{U_i\}_{i\in I}$ has a finite subcover, then X is compact.*

Proof. Let $\mathcal{T}$ be the collection of all open sets in X. We want to show that, if $\mathcal{S} \subseteq \mathcal{T}$ is a collection such that

$(*)$ $\forall S_0 \in \mathcal{S}$ finite, $\mathcal{S}_0$ does not cover X,

then $\mathcal{S}$ does not cover X. Let

$$\mathfrak{A} = \{S' \subseteq \mathcal{T} : S \subseteq S', S' \text{ satisfies } (*)\}$$

We claim that $\mathfrak{A}$ has a maximal element.

If $\mathfrak{A}_0 \subseteq \mathfrak{A}$ is linearly ordered, then $\bigcup_{\mathcal{S}' \subseteq A_0} \mathcal{S}'$ satisfies $(*)$ and is an upper bound. Then by Zorn's lemma, $\mathfrak{A}$ has a maximal element $\mathcal{S}_{\max}$. Note that

1. $\mathcal{S} \subseteq \mathcal{S}_{\max}$ and $\mathcal{S}_{\max}$ satisfies $(*)$ by construction.
2. If $U \in \mathcal{S}_{\max}, V \subseteq U$, then $V \in \mathcal{S}_{\max}$.
3. If $U, V \in \mathcal{S}_{\max}$, then $U \cup V \in \mathcal{S}_{\max}$.
4. If U is an open set such that $\forall V \in \mathcal{S}_{\max}, U \cup V \neq X$, then $U \in \mathcal{S}_{\max}$.

We will prove the last assertion. Let $\mathcal{S}'_{\max} = \mathcal{S}_{\max} \cup \{U\}$. By maximality, $\exists\{V_i\}_{i=1}^k \subseteq \mathcal{S}_{\max}$ such that $X = U \cup V_1 \cup \cdots \cup V_k$. But $V = V_1 \cup \cdots \cup V_k \in \mathcal{S}_{\max}$ by (*3*), so $U \cup V \neq X$. $\Rightarrow\Leftarrow$.

Suppose for the sake of contradiction $\mathcal{S}$ covers X, which means $\mathcal{S}_{\max}$ covers X. We claim that

$$\mathcal{U} = \{U_i : U_i \in S_{\max}\}$$

covers X. This will yield a contradiction, since $S_{\max}$ satisfies $(*)$, but a finite subset of the U_i covers X. Suppose otherwise, that $\mathcal{U}$ does not cover X. Fix $x \in X$. We know $\exists U \in S_{\max}$ such that $x \in U$. The U_i form a subbasis, so $U = \bigcup V_\alpha$, where each V_α is a finite intersection of U_i's (not necessarily in $\mathcal{S}_{\max}$. So $x \in V_\alpha$ for some α. $V_\alpha \in S_{\max}$ by (*2*).

$V_\alpha = U_1 \cap U_2 \cap \ldots \cap U_n$. We want to show that $\exists i : U_i \in S_{\max}$. Suppose not. Then by (*4*), $\exists W_i \in S_{\max}$ such that $U_i \cup W_i = X$. Then

$$U_1 \cap \cdots \cap U_n \cup W_1 \cup \cdots \cup W_n = X$$

But since $U_1 \cap \ldots \cap U_n = V_\alpha \in \mathcal{S}_{\max}, W_i \in S_{\max}$, $\Rightarrow\Leftarrow$. Then $\forall x \in X, \exists i : U_i \ni x, U_i \subseteq \mathcal{S}_{\max}$. ∎

Definition 14.2. A topological space X is <u>regular</u> if

1. $\forall x \in X, \{x\}$ is closed.
2. If $x \in X, K \subseteq X$ closed, $x \notin K$, then $\exists U, V$, $U \overset{\circ}{\ni} x, V \overset{\circ}{\supseteq} K : U \cap V = \emptyset$.

Note that any regular space is Hausdorff.

Claim 14.3. *Any Hausdorff space satisfies 1 above.*

Proof. Let $x \in X$. $\forall y \in X, y \neq x, \exists V_y \overset{\circ}{\ni} y, V_y \not\ni x$. Then $\bigcup V_y = X - \{x\}$ is open, so $\{x\}$ is closed. ∎

Claim 14.4. *Any compact Hausdorff space is regular.*

Claim 14.5. *Any metric space (X, d) is regular.*

Proof. If $K \subseteq X$, define $d(x, K) := \inf\{d(x, y) : y \in K\}$. Note that $d(x, K) = 0$ iff $x \in \overline{K}$. If K is closed and $x \notin K$, then $d(x, K) > 0$. Set $\epsilon = \frac{d(x,K)}{2}$. Let $U = B_\epsilon(x)$, and let $V = \bigcup_{y\in K} B_\epsilon(y)$. $U \cap V = \emptyset$, else $\exists z \in X : d(z, x) < \epsilon$, $d(z, y) < \epsilon \Longrightarrow d(x, y) < 2\epsilon = d(x, K)$, $\Rightarrow\Leftarrow$. ∎

Claim 14.6. *X is regular iff 1 holds and $\forall x \in X, W \ni x$ open, there exists another open neighborhood $U \ni x$: $\overline{U} \subseteq W$.*

Proof. Suppose X is regular, and fix $x \in X, W \ni x$ open. The set $K = X - W$ is clearly closed. Then $\exists U, V$ open, $U \ni x, V \supseteq K$. We know that $\overline{U} \cap K = \emptyset$, since $\forall y \in K, V \ni y$ is a neighborhood disjoint from U. So $\overline{U} \subseteq W$.

Now suppose that $\forall x \in X, W \ni x$ open, there exists an open $U \ni x : \overline{U} \subseteq W$. Fix $x \in X, K \subseteq X$ closed, $x \notin K$. Let $W = X - K$. Take $U \ni x$ as described. Then $X - \overline{U} \supseteq K$ and is open and disjoint from $U \ni x$, so X is regular. ∎

Proposition 14.7. *Any product of regular spaces is regular.*

Proof. Let $X = \prod X_\alpha$, each X_α regular. Let $x \in X$, $W \subseteq X$ open, $W \ni x$. We can assume WLOG that W is a basic open set, $W = \prod W_\alpha$, where each $W_\alpha \subseteq X_\alpha$ open, $W_\alpha = X_\alpha$ for almost all α. $x \in X$ has image

$x_\alpha \in X_\alpha$ for each α, and $x_\alpha \in W_\alpha$. Since X_α is regular, $\exists U_\alpha \ni x_\alpha : \overline{U}_\alpha \subseteq W_\alpha$. Note that if $W_\alpha = X_\alpha$, we can choose $U_\alpha = X_\alpha$. Let $U = \prod U_\alpha$. Clearly, $U \ni x$ and is open.

$$\overline{U} \subseteq \prod \overline{U}_\alpha \subseteq W$$

where $\prod \overline{U}_\alpha$ is closed. ■

Proposition 14.8. *Any subspace of a regular space is regular.*

Proof. Let X be a regular space, $Y \subseteq X$ a subspace. Say $y \in Y, K \subseteq Y$ closed in Y, $y \notin K$. $\overline{K} \cap Y = K$. Then $y \notin \overline{K}$. Since X is regular, $\exists U, V \subseteq X$ open, $U \ni y, V \supseteq \overline{K} : U \cap V = \emptyset$. But $U \cap Y$ and $V \cap Y \subseteq Y$ are both open, we have $y \in U \cap Y$ and $K \supseteq V \cap Y$, and $(U \cap Y) \cap (V \cap Y) \supseteq U \cap V = \emptyset$. ■

Definition 14.9. Let X, X' be topological spaces with the same underlying set. We say the topology on X is <u>finer</u> than the topology on X' (or the topology on X' is <u>coarser</u> than the topology on X) if every open set in X' is open in X, or, equivalently, if $\mathrm{id} : X \to X'$ is continuous.

Example. The box topology on $\prod X_\alpha$ is finer than the product topology.

Observation 14.10. Suppose that X, X' be topological spaces with the same underlying set. If X' is Hausdorff, then X is Hausdorff. However, if X' is regular, X need not be regular.

Example. Let $X' = \mathbb{R}$ with the usual topology. Let $X = \mathbb{R}$ be a refinement such that

$$K = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$$

is closed. That is, take $Y \subseteq \mathbb{R}$ to be closed with respect to X iff $Y = Y_1 \cup (K \cap Y_2)$, where $Y_1 \subseteq Y_2$ are closed subsets of $\mathbb{R}$. Open sets have the form $U = U_1 \cap (U_2 - K), U_2 \subseteq U_1$ for U_1, U_2 open. Then any open set containing zero contains $\epsilon \neq \frac{1}{n}$ that, made sufficiently small, intersects any open set about K. But then $\{0\}, K$ cannot be separated by disjoint open sets.

Proof. Left as exercise. ■

Lecture 15 — 10/6/10

Definition 15.1. A topological space X is <u>normal</u> if

1. $\forall x \in X, \{x\}$ is closed.
2. If $A, B \subseteq X$ are closed and $A \cap B = \emptyset$, then $\exists U \overset{\circ}{\supseteq} A, V \overset{\circ}{\supseteq} B : U \cap V = \emptyset$.

Observation 15.2. Any normal space is regular and Hausdorff. Any compact Hausdorff space is normal.

Claim 15.3. *Any metric space (X, d) is normal.*

Proof. Suppose $A, B \subseteq X$ are disjoint closed sets. Then $\forall a \in A, \exists \epsilon_a : B_{\epsilon_a}(a) \cap B = \emptyset$. Similarly, $\forall b \in B, \exists \epsilon_b : B_{\epsilon_b}(b) \cap A = \emptyset$. Let

$$U = \bigcup_{a \in A} B_{\frac{\epsilon_a}{2}}(a) \supseteq A \qquad V = \bigcup_{b \in B} B_{\frac{\epsilon_b}{2}}(b) \supseteq B$$

We want to show that $U \cap V = \emptyset$. Suppose otherwise, that $\exists x \in U \cap V$. Then $\exists a \in A : d(x, a) < \frac{\epsilon_a}{2}$ and $\exists b \in B : d(x, b) < \frac{\epsilon_b}{2}$. By the triangle inequality, $d(a, b) \leq d(a, x) + d(x, b) < \frac{\epsilon_a + \epsilon_b}{2}$. If $\epsilon_a \geq \epsilon_b$, $d(a, b) < \frac{\epsilon_a + \epsilon_b}{2} \leq \epsilon_a$. $\Rightarrow\Leftarrow$. Similarly, if $\epsilon_b \geq \epsilon_a$, $d(a, b) < \frac{\epsilon_a + \epsilon_b}{2} \leq \epsilon_b$. $\Rightarrow\Leftarrow$. ■

Claim 15.4. *A subspace of a normal space need not be normal. A product of normal spaces need not be normal.*

Proposition 15.5. *Let X be a second-countable topological space. Then X is regular iff X is normal.*

Proof. $\Longleftarrow$ True in general.

$\Longrightarrow$ Let $U_1, U_2, \ldots$ be a countable basis for X. Suppose that $A, B \subseteq X$ are disjoint closed subsets. By regularity, $\forall a \in A, \exists V_a \ni a$ open such that $\overline{V}_a \cap B = \emptyset$. WLOG, assume that $V_a \in \{U_1, U_2, \ldots\}$. Similarly, $\forall b \in B, \exists W_b \ni b$ open such that $\overline{W}_b \cap A = \emptyset$, and $W_b \in \{U_1, U_2, \ldots\}$.

Assume that $\{V_a\}_{a \in A} = \{V_1, V_2, \ldots\} \subseteq \{U_i\}$. Similarly, $\{W_b\}_{b \in B} = \{W_1, W_2, \ldots\} \subseteq \{U_i\}$. Define $V_n' = V_n - \bigcup_{j \leq n} \overline{W}_j$, and $W_n' = W_n - \bigcup_{j \leq n} \overline{V}_j$. Note that $V_n' \cap A = V_n \cap A$, so $V' = \bigcup_n V_n' \supseteq A$ (since the $\overline{W}_j$ are disjoint from A). Similarly, $W_n' \cap B = W_n \cap B$, so $W' = \bigcup_n W_n' \supseteq B$.

We claim now that $V' \cap W' = \emptyset$. Suppose otherwise. If $x \in V' \cap W'$, $x \in V_m' \cap W_n'$ for some m, n. If $m \leq n$, $x \in V_m' \supseteq V_m$. But $x \in W_n' = W_n - \bigcup_{j \leq n} \overline{V}_j$. $\Rightarrow\Leftarrow$. The case for $n \leq m$ is the same by symmetry. ■

Definition 15.6. An ordinal α is <u>countable</u> if it corresponds to a countable well-ordered set.

Note. There are uncountably many ordinals, and hence there exists a least uncountable ordinal ω_1.

Example. Let

$$A = \{\alpha : \alpha < \omega_1\} \qquad \overline{A} = \{\alpha : \alpha \leq \omega_1\} = A \cup \{\omega_1\}$$

Note that $\overline{A}$ has a topology, the <u>order topology</u>, with a basis of open sets

1. $(-\infty, \alpha) = \{\gamma : \gamma < \alpha\}$
2. $(\alpha, \infty) = \{\gamma : \gamma > \alpha\}$
3. $(\alpha, \beta) = \{\gamma : \alpha < \gamma < \beta\}$

We claim that there does not exist a sequence $\alpha_0, \alpha_1, \ldots \in A$ converging to $\omega_1 \in \overline{A}$. Suppose otherwise. Every open set containing ω_1 contains (α, ∞) for some $\alpha < \omega_1$. Then each α_i corresponds to a well-ordered set P_i. $\coprod P_i$ has order type $\alpha \geq \alpha_i, \alpha < \omega_1$.

Claim 15.7. *$\overline{A}$ is compact. Both A and $\overline{A}$ are normal.*

Claim 15.8. *$A \times \overline{A}$ is not normal. This shows that a product of normal spaces need not be normal. Moreover, since $\overline{A} \times \overline{A}$ is a compact Hausdorff space, and $A \times \overline{A} \subseteq \overline{A} \times \overline{A}$, this shows that a subspace need not be normal. We also see that a regular space need not be normal.*

Proof. Let

$$K = \{(\alpha, \alpha) : \alpha \in A\} \qquad K' = A \times \{\omega_1\}$$

K' is closed since it is the inverse image of the projection onto a point. $K = (A \times \overline{A}) \cap \Delta_{\overline{A} \times \overline{A}}$ the diagonal of $\overline{A} \times \overline{A}$, which is closed by Hausdorffness. Hence K is also closed.

Suppose that $U \supseteq K, U' \supseteq K'$ exist that separate K, K'. $\forall \alpha < \omega_1$, we claim that there is a least ordinal $T(\alpha)$ such that $T(\alpha) > \alpha, T(\alpha) < \omega_1$ and $(\alpha, T(\alpha)) \notin U$. By well-ordering, we need only show that $T(\alpha)$ exists. Suppose otherwise. Then $U \supseteq \{\alpha\} \times \{\beta \in A : \beta \geq \alpha\}$. $U' \supseteq K' \ni (\alpha, \omega_1) \Longrightarrow U \supseteq \{\alpha\} \times (\gamma, \infty)$ for some $\gamma < \omega_1$ (by the definitions of the product topology and the basis for the order topology). $\Rightarrow\Leftarrow$.

Let $\alpha_0 = 0, \alpha_1 = T(\alpha_0), \alpha_2 = T(\alpha_1)$. It is clear that $\alpha_0 < \alpha_1 < \cdots$. This sequence has a least upper bound $\beta < \omega_1$. We claim that $(\beta, \beta) \in U \cap (X - U)$. We know $(\beta, \beta) \in K \subseteq U$. But $X - U$ is closed and (β, β) is the limit of the sequence of points $(\alpha_0, \alpha_1), (\alpha_1, \alpha_2), \ldots$. But each $(\alpha_i, \alpha_{i+1}) \notin U$. Since $X - U$ closed, $(\beta, \beta) \in X - U$. $\Rightarrow\Leftarrow$. ∎

Lecture 16 — 10/8/10

Let X be a normal space, $A, B \subseteq X$ closed. One way to separate A from B is to choose a continuous function $f : X \to [0, 1]$ such that $\forall a \in A, b \in B$, $f(a) = 0$, $f(b) = 1$. Then $U = \{x \in X : f(x) < \frac{1}{2}\}$ and $V = \{x \in X : f(x) > \frac{1}{2}\}$ separate A and B.

Lemma 16.1 (Urysohn's Lemma)**.** *X is normal iff $\forall A, B \subseteq X$ closed, $A \cap B = \emptyset$, $\exists f : X \to [0, 1] : \forall a \in A$, $b \in B, f(a) = 0, f(b) = 1$.*

Proof. Let

$$Q = \{q \in \mathbb{Q} : 0 \leq q \leq 1\}$$

We can enumerate Q: $q_0 = 0, q_1 = 1, q_2, \ldots$. We will first construct, for each $q \in Q$, an open set $U_q \subseteq X$ such that

1. $A \subseteq U_0$
2. $B \cap U_1 = \emptyset$.
3. If $p < q, \overline{U}_p \subseteq U_q$.

and then construct our function $f : X \to [0, 1]$.

X is normal, so $\exists U_0 \overset{\circ}{\supseteq} A : \overline{U}_0 \cap B = \emptyset$. Define $U_1 = X - B$. We have satisfied *(1)* and *(2)*, and U_0, U_1 satisfy *(3)*. Now construct U_{q_n} by induction on n. Assume that $\forall i < n, n \geq 2$, U_{q_i} is chosen. We can choose $q_-, q_+ \in \{q_0, \ldots, q_{n-1}\} := Q_n$ such that $q_- < q_n < q_+$, where q_- is the greatest lower bound and q_+ is the least upper bound of q_n in Q_n. Then $\overline{U}_{q_-} \subseteq U_{q_+}$. $\overline{U}_{q_0}$ does not interesect $X - U_{q_+}$. Since X is normal, $\exists V, W \subseteq X$ such that $V \overset{\circ}{\supseteq} \overline{U}_{q_-}, W \overset{\circ}{\supseteq} X - U_{q_+}$ and $V \cap W = \emptyset$.

Set $U_{q_n} = V$. We claim that $\overline{U}_{q_n} = \overline{V} \subseteq U_{q_+}$. But by construction,

$$V \subseteq \overline{V} \subseteq X - W \subseteq U_{q_+}$$

Now we want to construct f. Define

$$f(x) = \begin{cases} \inf\{q \in Q : x \in \overline{U}_q\} & \text{if nonempty} \\ 1 & \text{otherwise} \end{cases}$$

It is clear that $\forall a \in A, b \in B, f(a) = 0, f(b) = 1$ by construction.

We want now to show that f is continuous. The space $[0, 1] \subseteq \mathbb{R}$ has a basis consisting of open intervals $(r, t) : r, t \in \mathbb{R}$. It suffices to show that $f^{-1}(r, t)$ is open in X. But $(r, t) = (r, \infty) \cap (-\infty, t)$, so it suffices to show that $f^{-1}(r, \infty)$ and $f^{-1}(-\infty, t)$ are open. Call

$$X_r = f^{-1}(r, \infty) = \{x \in X : f(x) > r\}$$

If $r < 0, X_r = X$ is open, and if $r \geq 1, X_r = \emptyset$ is open. Assume $0 \leq r < 1$. Then

$$X_r = \left\{x \in X : \exists q > r : q \leq \{p : x \in \overline{U}_p\}\right\}$$

We know $\exists q' \in Q : r < q' < q$. It follows that

$$\begin{aligned} X_r &= \{x \in X : \exists q' > r, x \notin \overline{U}_{q'}\} \\ &= \bigcup_{\substack{q \in Q \\ q > r}} (X - \overline{U}_q) \\ &= f^{-1}(r, \infty) \end{aligned}$$

is open.

Now we want to show that

$$X_t = f^{-1}(-\infty, t) = \{x \in X : f(x) < t\}$$

is open. If $t > 1, X_t = X$ is open, and if $t \leq 0, X_t = \emptyset$ is open. Assume $0 < t < 1$. Then

$$X_t = \left\{x \in X : \exists q < t : q \leq \{p \in Q : x \in \overline{U}_p\} \neq \emptyset\right\}$$

We know $\exists q' \in Q : q < q' < t$. It follows that

$$X_t = \{x \in X : \exists q' < t, x \in \overline{U}_{q'}\}$$

Since $q' < t, \exists p \in Q : q' < p < t$. Then $x \in \overline{U}_{q'} \implies x \in U_p$. So

$$\begin{aligned} X_t &= \{x \in X : \exists q < t, x \in U_q\} \\ &= \bigcup_{\substack{q \in Q \\ q < t}} U_q \\ &= f^{-1}(-\infty, t) \end{aligned}$$

is open. ■

Corollary 16.2 (Tietze Extension Theorem)**.** *Let X be a normal space, $Y \subseteq X$ a closed subspace. Let $f_Y : Y \to [a, b] \subseteq \mathbb{R}$ be a continuous function. Then f_Y extends to a continuous function $f : X \to [a, b]$.*

Proof. WLOG, take $f_Y : Y \to [-b, b]$. Consider the sets $A = f_Y^{-1}[-b, -\frac{1}{3}b]$ and let $B = f_Y^{-1}[\frac{1}{3}b, b]$. A and B are closed in X. By Urysohn's Lemma, $\exists g_0 : X \to [-\frac{1}{3}b, \frac{1}{3}b]$ continuous such that $g_0(x) = -\frac{1}{3}b$ if $x \in A$, $g_0(x) = \frac{1}{3}b$ if $x \in B$. Note that if $y \in Y$, $|f_Y(y) - g_0(y)| \leq \frac{2}{3}b$.

Now define a function $f_Y^1 : Y \to [-\frac{2}{3}b, \frac{2}{3}b]$ given by $f_Y^1(y) = f_Y(y) - g_0(y)$. By the same reasoning, we get a function $g_1 : X \to [-\frac{2}{9}b, \frac{2}{9}b]$ such that $|f_Y^1(y) - g_1(y)| \leq \frac{4}{9}b$. Continuing this process, we get a sequence of functions $g_i : X \to [-\frac{2^i}{3^{i+1}}b, \frac{2^i}{3^{i+1}}b]$. Now define $f(x) = \sum_i g_i(x)$.

We claim that f is well-defined and continuous (proof left as exercise). By construction,

$$|f_Y(y) - \sum_{i \leq n} g_i(x)| < \left(\frac{2}{3}\right)^n b$$

so $f_Y = f|_Y$, as desired. ■

Lecture 17 — 10/13/10

Theorem 17.1 (Tietze Extension Theorem)**.** *If X is normal, $Y \subseteq X$ a closed subset, then any continuous map $Y \to [0, 1]$ extends to X.*

Claim 17.2. *The Tietze Extension Theorem implies Urysohn's Lemma (if X is normal, $A, B \subseteq X$ closed and disjoint, then $\exists f : X \to [0, 1]$ such that $f|_A = 0, f|_B = 1$).*

Proof. Note that $A \cup B$ is closed; simply extend the function $A \cup B \to [0, 1]$ to a function $X \to [0, 1]$. ■

Theorem 17.3 (Urysohn's Metrization Theorem)**.** *Let X be a second-countable topological space. TFAE:*

1. *X is metrizable.*
2. *X is normal.*
3. *X is regular.*
4. *X is homeomorphic to a subspace of $\prod_{n \in \mathbb{N}} [0, 1]$.*

Proof.

1 ⇒ 2. True in general.

2 ⇒ 3. True in general.

3 ⇒ 2. True in second-countable spaces.

4 ⇒ 1. True in general (from HW).

2 ⇒ 4. By second-countability, there exists a countable basis $\{U_i\}$ for the topology on X. For each $i, j \in \mathbb{N}$ such that $\overline{U}_i \subseteq U_j$ (which implies that $\overline{U}_i \cap (X - U_j) = \emptyset$), there exists a continuous function $\lambda_{i,j} : X \to [0, 1]$ such that $\lambda_{i,j}|_{\overline{U}_i} \equiv 0$ and $\lambda_{i,j}|_{X - U_j} \equiv 1$. So we have a countable collection of continuous functions $\lambda_{i,j} : X \to [0, 1]$. These yield a continuous map

$$\lambda : X \to \prod_{\substack{i,j: \\ \overline{U}_i \subseteq U_j}} [0, 1]$$

We claim that λ is injective and is a homeomorphism from X to $\lambda(X) \subseteq \prod [0, 1]$.

Let $x, y \in X : x \neq y$. To show injectivity, it suffices to show that $\lambda_{i,j}(x) \neq \lambda_{i,j}(y)$ for some $\lambda_{i,j}$. Since X is regular, we can choose $U \ni x$ open, $\overline{U} \not\ni y$. Then $x \in U_j \subseteq U$ for some basic open set U_j. Then $y \in X - \overline{U}_j \subseteq X - U_j$. Since $X - U_j$ is closed and X is regular, $\exists V \ni x$ open such that $\overline{V} \subseteq U_j$. Assume WLOG that $V = U_i$ for some basic open set U_i. Then we have $\overline{U}_i \subseteq U_j$, and for this i, j, we have $\lambda_{i,j}(x) = 0, \lambda_{i,j}(y) = 1$. So λ is injective.

Regard $\lambda(X) \subseteq \prod [0, 1]$ as a topological space. We know that $\lambda : X \to \lambda(X)$ is continuous and bijective. It remains to be shown that $\lambda^{-1} : \lambda(X) \to X$ is continuous. Note that

$$x \in U_i \subseteq \lambda_{i,j}^{-1}[0, 1) \subseteq U_j \subseteq U$$

Consider that

$$U \subseteq \bigcup_{\substack{i,j: \\ \overline{U}_i \subseteq U_j \subseteq U}} \lambda_{i,j}^{-1}[0, 1) \subseteq U$$

We want to show that $\lambda(U)$ is open in $\lambda(X)$. But

$$\lambda(U) = \bigcup_{\substack{i,j: \\ \overline{U}_i \subseteq U_j \subseteq U}} \lambda\Big(\lambda_{i,j}^{-1}[0,1)\Big)$$

It suffices to show that $\lambda(\lambda_{i,j}^{-1}[0,1))$ is open in $\lambda(X)$. We claim that

$$\lambda(\lambda_{i,j}^{-1}[0,1)) = \lambda(X) \cap \pi_{i,j}^{-1}[0,1)$$

Let $y \in \lambda(X)$; then $\exists x \in X : y = \lambda(x)$. Then we know that $y \in \lambda(\lambda_{i,j}^{-1}[0,1))$ iff $\lambda_{i,j}(x) \in [0,1)$. This is true iff

$$\pi_{i,j}(y) = \pi_{i,j}(\lambda(x)) = \lambda_{i,j}(x) \in [0,1)$$

Hence, our claim holds. Since $[0,1)$ is open in $[0,1]$, this means that $\lambda(U)$ is a union of open sets and hence is open. So λ is a homeomorphism, as desired. ∎

Definition 17.4. Let $f : X \to Y$ be a map. The <u>support</u> of f is

$$\operatorname{supp}(f) = \{y \in Y : f(x) \neq 0\}$$

Definition 17.5. A topological space X is <u>completely regular</u> if

1. $\forall x \in X, \{x\} = \overline{\{x\}}$.
2. $\forall x \in X, \forall K \subseteq X$ closed, $K \not\ni x$, there is a continuous function $f : X \to [0,1]$ such that $f(x) = 1$, $f|_K \equiv 0$.

Equivalently, our second condition states that $\forall U \ni x$ open, $\exists f : X \to [0,1]$ continuous such that $f(x) = 1$, $\operatorname{supp}(f) \subseteq U$ since $X - U$ is closed.

Note. All normal spaces are completely regular, and all completely regular spaces are regular. Note also that if X is second-countable, then normality, regularity, and complete regularity are equivalent.

Proposition 17.6. *Any product of completely regular spaces is completely regular.*

Proof. Let $X = \prod_{i \in I} X_i$ each X_i completely regular. Take $x \in X, x = (x_i)_{i \in I}, x_i \in X_i$ and let $U \subseteq X, U \ni x$ open. WLOG, assume U is basic, so $U_i = X_i$ for all $i \notin I_0 \subseteq I$ a finite set. $\forall i \in I$, we can choose a continuous function $f_i : X_i \to [0,1]$ such that $f_i(x_i) = 1$ and $\operatorname{supp}(f_i) \subseteq U_i$. For $i \notin I_0$, we can take $f_i \equiv 1$. Then define $f : X \to [0,1]$ by

$$f(y) = \prod f_i(y_i)$$

f is well-defined and continuous, and we easily see that $f(x) = 1$. Moreover,

$$\begin{aligned}\operatorname{supp}(f) &= \{y : f(y) \neq 0\} \\ &= \{y : \forall i \in I, y_i \in \operatorname{supp}(f_i)\} \\ &\subseteq U\end{aligned}$$

so X is completely regular, as desired. ∎

Lecture 18 — 10/15/10

Proposition 18.1. *If X is completely regular, any subspace of X is completely regular.*

Proof. Let $Y \subseteq X$ be a subspace. Let $x \in Y$ and $U \subseteq Y, U \ni x$ an open neighborhood. We want a continuous function $f : Y \to [0,1]$ such that $f(x) = 1, \operatorname{supp}(f) \subseteq U$. By definition, $U = V \cap Y$ for some $V \subseteq X$ open. By complete regularity, $\exists g : X \to [0,1]$ a continuous function, $g(x) = 1, \operatorname{supp}(g) \subseteq V$. Define $f : Y \to [0,1]$ by $f(y) = g(y), f = g|_Y$. ∎

Theorem 18.2. *Let X be a topological space. TFAE:*

1. *X is completely regular.*
2. *X is homeomorphic to a subspace of $\prod_{s \in S}[0,1]$.*
3. *X is homeomorphic to a subspace of some compact Hausdorff space.*

Proof.

2 ⇒ 3. Hausdorffness is known, and compactness is given by Tychonoff's Theorem.

3 ⇒ 1. All compact Hausdorff spaces are normal, and hence completely regular.

1 ⇒ 2. Assume that X is completely regular. Choose an index set

$$S = \{f_s : X \to [0,1]\}$$

We get a canonical map $\varphi : X \to \prod_{s \in S}[0,1]$, with s-coordinate f_s, given by $x \mapsto (f_s(x))_{s \in S}$. We claim that φ is injective and defines a homeomorphism $X \to \varphi(X) \subseteq \prod[0,1]$.

Suppose $x, y \in X : x \neq y$. We want to show that $\exists s \in S : f_s(x) \neq f_s(y)$. Since $x \neq y$, we know that $X - \{y\} \ni x$ and is open. Then $\exists f : X \to [0,1]$ such that $f(x) = 1, \operatorname{supp}(f) \subseteq X - \{y\} \iff f(y) = 0$. Hence, φ is injective.

Let

$$\mathcal{U} = \{\varphi^{-1}(U) : U \overset{\circ}{\subseteq} \prod_{s \in S}[0,1]\}$$

We know that $\mathcal{U}$ is a collection of open subsets of X; we want it to be the entire topology on X. Let $V \subseteq X$ be any open set. For each $x \in V$, we can choose $f_x : X \to [0,1], f_x(x) = 1, \operatorname{supp}(f_x) \subseteq V$. Let $V_x = \operatorname{supp}(f_x) \ni x$. Then $\bigcup V_x = V$. But

$$\begin{aligned} V_x &= \{y \in X : f_x(y) > 0\} \\ &= \varphi^{-1}\left(\Big(\prod_{\substack{s \in S \\ f_s \neq f_x}} [0,1]\Big) \times (0,1]\right) \end{aligned}$$

Let us write

$$U_x := \prod_{\substack{s \in S \\ f_s \neq f_x}} [0,1]$$

so $V_x = \varphi^{-1}\big(U_x \times (0,1]\big)$. Then

$$\begin{aligned} V = \bigcup V_x &= \bigcup \varphi^{-1}(U_x \times (0,1]) \\ &= \varphi^{-1} \bigcup (U_x \times (0,1]) \\ &\in \mathcal{U} \end{aligned}$$

Thus, φ is a homeomorphism, as desired. ■

Corollary 18.3. *Let X be a topological space. Then X is a compact Hausdorff space iff X is homeomorphic to a closed subspace of $\prod[0,1]$.*

Proof. We know that $\exists \varphi : X \to \prod[0,1]$ a homeomorphism by Theorem 18.2. $\varphi(X)$ is homeomorphic to X, and hence $\varphi(X)$ is compact. By Hausdorffness, $\varphi(X)$ is closed. ■

Definition 18.4. Let X be a topological space. A compactification of X is a map $\varphi : X \to \overline{X}$ such that

1. φ is a homeomorphism onto $\varphi(X)$.
2. $\overline{X}$ is compact Hausdorff.

We can, in some sense, consider φ an inclusion map $X \hookrightarrow \overline{X}$.

Theorem 18.5. *There exists a compactification of X iff X is completely regular.* ■

Example. If X is compact Hausdorff, $\mathrm{id} : X \to X$ is a compactification of X.

Example. $\mathbb{R}^n$ is not compact but is homeomorphic to $S^n - \{0\}$. This is called the one-point compactification.

Observation 18.6. Let $\varphi : X \to Y$ be a compactification of X, $\varphi : X \to \overline{\varphi(X)} \subseteq Y$. Then $X \to \overline{\varphi(X)}$ is also a compactification. We might require in our definition that $\varphi(X)$ be dense in $\overline{X}$.

Definition 18.7. Let X be any completely regular space, $\varphi : X \to \prod_{s \in S}[0,1]$ a homeomorphism. Then

$$\hat{X} := \overline{\varphi(X)} \subseteq \prod_{s \in S}[0,1]$$

is a compactification of X, called the Stone-Čech compactification of X.

Theorem 18.8. *Let X be any completely regular space, let $f : X \to Y$ be any continuous map into a compact Hausdorff space. Then f lifts uniquely to a map $\hat{f} : \hat{X} \to Y$.*

$$\begin{tikzcd} X \arrow[r, hook, "f"] \arrow[d, hook, "\varphi"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \\ \hat{X} \arrow[ur, "\hat{f}"'] & \end{tikzcd}$$

Proof. WLOG, Y is a closed subset $Y \subseteq Y' = \prod_{s \in S}[0,1]$. Then $f : X \to Y \subseteq Y'$. Suppose that f does indeed lift to a map $\hat{f} : \hat{X} \to Y'$. We claim that f factors through $\hat{X}$; that is, $\hat{f}^{-1}(Y) = \hat{X}$. Y is closed and $\hat{f}$ is continuous, so

$$X \cong \varphi(X) \subseteq \hat{f}^{-1}(Y) \subseteq \hat{X}$$

is closed. Then $\hat{f}^{-1}(Y) = \hat{X}$.

Hence, it suffices to assume that $Y = \prod[0,1]$. By components, we can assume WLOG that $Y = [0,1]$. First, we will show the existence of our desired function $\hat{f} : \hat{X} \to Y$. We know that our function $f : X \to Y = [0,1]$ is such that $f = f_s$ for some $s \in S$. Then

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[drr, "f_s = f"'] & \hat{X} \arrow[dr, "\hat{f}"] & \subseteq & \prod_{s \in S}[0,1] \arrow[dl, "\pi_s"] \\ & & {[0,1]} & \end{tikzcd}$$

reveals a map $\hat{f} = \pi_s \circ \varphi$ from $\hat{X} \to [0,1]$.

Now we will show that $\hat{f}$ is unique. Say we have $\hat{f}, \hat{f}' : \hat{X} \to [0,1]$ such that $\hat{f}|_X = f = \hat{f}'|_X$. We want to show that $\hat{f} = \hat{f}'$. Let

$$Z := \{x \in \hat{X} : \hat{f}(x) = \hat{f}'(x)\} \subseteq \hat{X}$$

Note that, taking

$$g = (\hat{f}, \hat{f}') : \hat{X} \to [0,1] \times [0,1]$$

we get $Z = g^{-1}(\Delta)$, the inverse image of the diagonal, and hence Z is closed. We want to show that $Z = \hat{X}$. Since $\hat{f}|_X = \hat{f}'|_X$, we know that $\varphi(X) \subseteq Z$, and since Z is closed, we get

$$Z \subseteq \hat{X} = \overline{\varphi(X)} \subseteq Z$$

and hence $Z = \hat{X}$, so $\hat{f}$ is unique, as desired. ■

Lecture 19 — 10/18/10

Definition 19.1. A Hausdorff space X is locally compact if $\forall x \in X, \exists U \ni x$ open such that $\overline{U} \subseteq X$ is compact.

Example. Any compact Hausdorff space is locally compact. $\mathbb{R}^n$ is locally compact (take the closure of open balls).

Claim 19.2. *Let X be a compact Hausdorff space, $U \subseteq X$ an open subset. Then U is locally compact.*

Proof. Let $x \in U$. Since X is compact Hausdorff, it is regular. Then $\exists V \subseteq X$ open, $V \ni x$, $\overline{V} \subseteq U$. $\overline{V}$ is compact (since it is a closed subset of X). ∎

Theorem 19.3. *Let X be a Hausdorff space. TFAE:*

1. *X is locally compact.*
2. *There exists a compact Hausdorff space Y such that X is homeomorphic to an open subset of Y.*
3. *There exists a compact Hausdorff space Y and a point $y \in Y$ such that $X \cong Y - \{y\}$.*

Proof.

3 ⇒ 2. $Y - \{y\}$ is open since $\{y\}$ is closed.

2 ⇒ 1. True by the above claim.

1 ⇒ 3. Assume that X is locally compact, and define

$$\overline{X} = X \cup \{\infty\}$$

We want a topology on $\overline{X}$ which agrees with the topology on X such that $\overline{X}$ is compact Hausdorff. Let a set $U \subseteq \overline{X}$ be open if either $U \subseteq X$ is open, or $U = V \cup \{\infty\}$ where $V \subseteq X$ is open and $X - V$ is compact. We claim that this is indeed a topology.

Clearly, $\emptyset$ and $\overline{X} = X \cup \{\infty\}$ are open. Let U, U' be open. If $\infty \notin U \cap U'$, then $U \cap U'$ is open in X and hence in $\overline{X}'$. Suppose $\infty \in U, U'$. Then $U = V \cup \{\infty\}$ and $U' = V' \cup \{\infty\}$, so $U \cap U' = (V \cap V') \cup \{\infty\}$. But $X - (V \cap V') = (X - V) \cup (X - V')$ is compact (since finite unions of compact sets are compact; left as exercise), so $U \cap U'$ is open.

Now let $U = \bigcup U_\alpha$, where the U_α are open. If $\infty \notin U_\alpha, \forall \alpha$, then U is open in $\overline{X}$ by openness in X. Suppose that $\infty \in U_\alpha$ for some α. Then $U \ni \infty$. Let us write $V_\beta = X \cap U_\beta$. So $U = \bigcup V_\beta \cup \{\infty\}$. We want to show that $X - \bigcup V_\beta$ is compact. But $X - \bigcup V_\beta \subseteq X - V_\alpha$ which is compact, so $X - \bigcup V_\beta$ is compact since it is closed.

Now we want to show that $\overline{X}$ is Hausdorff. Choose $x, y \in \overline{X} : x \neq y$. If $x, y \in X$, this follows from X Hausdorff. Suppose now that $x \in X, y = \infty$. We want $U \ni x, U' \ni y$ such that $U \cap U' = \emptyset$. Then $U' = V \cup \{\infty\}$ where $V \subseteq X$ open, $X - V$ compact, and moreover $U \cap V = \emptyset$. By local compactness, we may choose U such that $\overline{U}$ is compact, and define $V = X - \overline{U}$. Then $\overline{X}$ is Hausdorff.

Finally, we want to show that $\overline{X}$ is compact. Let $\{U_\alpha\}$ be an open cover of $\overline{X}$. Then $\infty \in U_\alpha$ for some α, so $U_\alpha = V_\alpha \cup \{\infty\}, X - V_\alpha$ compact. Again, let us write $V_\beta = X \cap U_\beta$. Then $\bigcup_{\beta \neq \alpha} V_\beta \supseteq X - V_\alpha$. By compactness, $X - V_\alpha \subseteq \bigcup_{\beta \in I} V_\beta$ where I is finite. So $\overline{X} = U_\alpha \cup \bigcup_{\beta \in I} U_\beta$. ∎

Definition 19.4. The one-point compactification of X is $\overline{X} = X \cup \{\infty\}$.

Remark. Let X, Y be topological spaces. Let

$$\mathrm{Map}(X, Y) = \{f : X \to Y, f \text{ continuous}\}$$

We want to put a nice topology on $\mathrm{Map}(X, Y)$.

Example. Say X has the discrete topology. Then every map $X \to Y$ is continuous, so

$$\mathrm{Map}(X, Y) = Y^X = \prod_{x \in X} Y$$

has the product topology.

Definition 19.5. Giving a map $Z \to \mathrm{Map}(X, Y)$ is the same as giving a map $\mu : X \times Z \to Y$ such that $\forall z \in Z$, the map $x \mapsto \mu(x, z)$ is continuous. We say that the map $Z \to \mathrm{Map}(X, Y)$ is induced by μ.

Remark. We want a topology on $\mathrm{Map}(X, Y)$ such that giving a *continuous* map $Z \to \mathrm{Map}(X, Y)$ is the same as giving a *continuous* map $\mu : X \times Z \to Y$.

Claim 19.6. *A topology on* $\mathrm{Map}(X, Y)$ *satisfying the above property is unique.*

Proof. Suppose $\mathrm{Map}(X, Y)$ can be equipped with two different topologies, $\mathcal{T}, \mathcal{T}'$ satisfying the given property; we will abbreviate these different topologies by writing $\mathrm{Map}(X, Y)$ and $\mathrm{Map}'(X, Y)$. Consider the identity map

$$\mathrm{id} : \mathrm{Map}(X, Y) \to \mathrm{Map}(X, Y)$$

Then the evaluation map

$$X \times \mathrm{Map}(X, Y) \to Y$$

is continous, where $\mathrm{Map}(X, Y)$ is read as Z above. By the same property, we get a continuous map $\mathrm{Map}(X, Y) \to \mathrm{Map}'(X, Y)$, which is the identity map. Then $\mathrm{Map}(X, Y) \cong \mathrm{Map}'(X, Y)$ by the identity map; hence, they are identical. ∎

Remark. It is generally not possible to put such a topology on $\mathrm{Map}(X, Y)$.

Proposition 19.7. *If X is a locally compact Hausdorff space, then such a topology exists on* $\mathrm{Map}(X,Y)$ *for any topological space Y.*

Remark. Let (Y,d) be a metric space. We might try to define the distance between two maps as the maximum of the distance between those maps evaluated at any point. But this distance does not necessarily exist if X is not compact. In this case, we will try to do this "separately" for all compact subsets of X.

Lecture 20 — 10/20/10

Definition 20.1. Consider the set $\mathrm{Map}(X,Y)$. We define the compact-open topology on $\mathrm{Map}(X,Y)$ to be the topology generated by the subbasis

$$\{f \in \mathrm{Map}(X,Y) : f(K) \subseteq U\}$$

taken over all compact $K \subseteq X$, and all open $U \subseteq Y$. Our basis is therefore given by

$$\{f \in \mathrm{Map}(X,Y) : f(K_1) \subseteq U_1, \ldots, f(K_n) \subseteq U_n\}$$

Proposition 20.2. *If X is locally compact, the evaluation map*

$$\begin{aligned} \mathrm{ev} : X \times \mathrm{Map}(X,Y) &\longrightarrow Y \\ (x,f) &\longmapsto f(x) \end{aligned}$$

is continuous.

Proof. Say $U \subseteq Y$ is open. We want to show that $\mathrm{ev}^{-1}(U) \subseteq X \times \mathrm{Map}(X,Y)$ is open. Note that

$$\mathrm{ev}^{-1}(U) = \{(x,f) : f(x) \in U\}$$

Say $(x_0, f_0) \in \mathrm{ev}^{-1}(U)$. Then $f_0(x_0) \in U$. We want an open set $V \subseteq X, V \ni x_0$ and an open set $W \subseteq \mathrm{Map}(X,Y)$, $W \ni f_0$ such that $V \times W \subseteq \mathrm{ev}^{-1}(U)$. That is, we want $f(x) \in U, \forall x \in V, f \in W$. Since f_0 is continuous, $f_0^{-1}(U) \ni x_0$ is an open subset of X. Since X is locally compact, X is regular. Then $\exists V \ni x$ open such that $\overline{V} \subseteq f_0^{-1}(U)$. WLOG, assume that $\overline{V}$ is compact. Let

$$W = \{f \in \mathrm{Map}(X,Y) : f(\overline{V}) \subseteq U\}$$

which is open by its definition. Then $\mathrm{ev}(V \times W) \subseteq U$. ∎

Note. For any topological space Z, a continuous map $g : Z \to \mathrm{Map}(X,Y)$ yields a continuous map

$$X \times Z \longrightarrow X \times \mathrm{Map}(X,Y) \xrightarrow{\mathrm{ev}} Y$$

which we call

$$\bar{g} : X \times Z \longrightarrow Y$$

Proposition 20.3. *Let X be a locally compact topological space. Then $g \mapsto \bar{g}$ is a bijection between continuous maps $Z \to \mathrm{Map}(X,Y)$ and continuous maps $X \times Z \to Y$.*

Proof. To construct the inverse, begin with a continuous map $\bar{g} : Z \times X \to Y$. For each $z \in Z$, let $\bar{g}_z : X \to Y$ be given by $\bar{g}_z(x) = \bar{g}(x,z)$. So $z \mapsto \bar{g}_z$ defines a map $g : Z \to \mathrm{Map}(X,Y)$. We want to show that g is continuous.

Let $U \subseteq Y$ be open, $K \subseteq X$ be compact. Let

$$V = \{f \in \mathrm{Map}(X,Y) : f(K) \subseteq U\}$$

We want to show that $g^{-1}(V)$ is open in Z.

$$g^{-1}(V) = \{z \in Z : \bar{g}(x,z) \in U, \forall x \in K\}$$

Say $z_0 \in g^{-1}(V)$. Then $\forall x \in K, \bar{g}(x,z_0) \in U$. So there exists a neighborhood of (x,z_0) belonging to $\bar{g}^{-1}(U)$ about every x. Then $\exists V_x \subseteq X, W_x \subseteq Z$ open such that $V_x \ni x, W_x \ni z_0$ and $\bar{g}(V_x \times W_x) \subseteq U$. Then

$$K \subseteq \bigcup_{x \in K} V_x$$

Since K is compact, we can choose a finite subcover

$$V_{x_1} \cup \cdots \cup V_{x_n} \supseteq K$$

Let

$$W = \bigcap_{i \leq n} W_{x_i}$$

Then

$$\begin{aligned} \bar{g}((V_{x_1} \cup \cdots \cup V_{x_n}) \times W) &= \bigcup \bar{g}(V_{x_i} \times W) \\ &\subseteq \bigcup \bar{g}(V_{x_i} \times W_{x_i}) \\ &\subseteq U \end{aligned}$$

So $\bar{g}(K \times W) \subseteq U$. So for every $z \in W$, we have $\bar{g}_z(K) \subseteq U$. Then $\forall z \in W, \bar{g}_z \in V$. Hence,

$$z_0 \in W \subseteq g^{-1}(V) \subseteq Z$$

so $g^{-1}(V)$ is open, as desired. ∎

Definition 20.4. Say Y is a metric space, X an arbitrary set. Let $f, g : X \to Y$. Define

$$\rho(f,g) = \sup_{x \in X} \Big\{ \min\big(1, d(f(x), g(x))\big) \Big\}$$

We call the topology induced by this metric the uniform topology.

Claim 20.5. *This is indeed a metric on Y^X, and hence on* $\mathrm{Map}(X,Y)$*, if X is equipped with any topology.*

Proof. Left as exercise. ∎

Observation 20.6. A sequence $f_1, f_2, \ldots \in \mathrm{Map}(X,Y)$ converges to f by uniform convergence with respect to the uniform topology:

$$(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall x \in X)(\forall n > N)(d(f_n(x), f(x)) < \epsilon)$$

Compare this with pointwise convergence:

$$(\forall \epsilon > 0)(\forall x \in X)(\exists N \in \mathbb{N})(\forall n > N)(d(f_n(x), f(x)) < \epsilon)$$

Example. For functions $[0,1] \to \mathbb{R}$, a uniform limit of continuous functions is always continuous, but a pointwise limit need not be.

Remark. Let X be a topological space, Y a metric space. $\mathrm{Map}(X,Y)$ can be equipped either with the compact open topology or with the metric topology; however, these two topologies are not the same in general.

Proposition 20.7. *Let X be a compact topological space, Y a metric space. Then the compact-open topology ($\mathcal{T}_{KU}$) and the uniform topology ($\mathcal{T}_d$) on $\mathrm{Map}(X,Y)$ coincide.*

Proof. We claim that the identity maps between these topological spaces are continuous. We will show first that

$$\mathrm{id} : \mathrm{Map}_d(X,Y) \to \mathrm{Map}_{KU}(X,Y)$$

is continuous. For this direction, we need only assume that X is locally compact. By Proposition 20.3, it suffices to give a continuous map $X \times \mathrm{Map}_d(X,Y) \to Y$. Consider the evaluation map; we will show it is continuous.

Say $U \subseteq Y$ open, $x_0 \in X$, $f_0 \in \mathrm{Map}(X,Y)$, and $f_0(x_0) \in U$. We want to find open sets $V \subseteq X, V \ni x$ and $W \subseteq \mathrm{Map}_d(X,Y), W \ni f_0$ such that $\mathrm{ev}(V \times W) \subseteq U$. Let $y_0 = f_0(x_0)$. Since $U \ni y_0$ is open, $\exists \epsilon : B_\epsilon(y_0) \subseteq U$. Since f_0 is continuous, $\exists V \subseteq X$ open, $V \ni x_0$ such that $f_0(V) \subseteq B_{\epsilon/2}(y_0)$. Let $W = B_{\epsilon/2}(f_0)$. Let $x \in V, f \in W$. Then

$$d(f(x), f_0(x_0)) \leq d(f(x), f_0(x)) + d(f_0(x), f_0(x_0)) < \epsilon$$

So $\forall x \in V, f \in W, f(x) \in B_\epsilon(y_0) \subseteq U$, as desired.

Now we want to show that

$$\mathrm{id} : \mathrm{Map}_{KU}(X,Y) \to \mathrm{Map}_d(X,Y)$$

is continuous. We want to show that if $f_0 \in \mathrm{Map}(X,Y)$, $\epsilon > 0$, then $B_\epsilon(f_0) \subseteq \mathrm{Map}_{KU}(X,Y)$ is open. WLOG, take $\epsilon < 1$. Say $f \in B_\epsilon(f_0)$. We want a neighborhood of $W \ni f$ open w.r.t. $\mathcal{T}_{KU}$ such that $W \subseteq B_\epsilon(f_0)$. By assumption, $\rho(f, f_0) < \epsilon$, so choose

$$\delta = \epsilon - \rho(f, f_0)$$

So $B_\delta(f) \subseteq B_\epsilon(f_0)$. Hence, WLOG, we can replace (f_0, ϵ) by (f, δ); that is, we may assume $f = f_0$.

For each $x \in X$, the set

$$U_x := f_0^{-1}\Big(B_{\frac{\epsilon}{2}}(f_0(x))\Big) \subseteq X$$

is open in X and contains x. These open sets cover X. Assuming X is compact, we can find a finite subcover given by $x_1, \ldots, x_n$:

$$U_{x_1} \cup \cdots \cup U_{x_n} \supseteq X$$

Let

$$W = \{f \in \mathrm{Map}(X,Y) : \forall i \leq n, f(\overline{U}_{x_i}) \subseteq B_{\frac{\epsilon}{2}}(f_0(x_i))\}$$

$W \subseteq \mathrm{Map}_{KU}(X,Y)$ is open because it is an intersection of finitely many basic open sets. We claim that $W \subseteq B_\epsilon(f_0)$; that is, if $f \in W$, then $\forall x \in X, d(f(x), f_0(x)) < \epsilon$. Then

$$d(f(x), f_0(x)) \leq \underbrace{d(f(x), f_0(x_i))}_{f \in W} + \underbrace{d(f_0(x_i), f_0(x))}_{\exists i : x \in U_{x_i}} < \epsilon$$

for every $x \in X$, and therefore

$$\rho(f, f_0) < \epsilon$$

as desired. ∎

Lecture 21 — 10/22/10

Recall that if X is compact Hausdorff and (Y,d) is a metric space, then $\mathrm{Map}(X,Y)$ w.r.t. the compact-open topology is also a metric space w.r.t. the uniform metric.

Proposition 21.1. *If X is not compact, then convergence in $\mathrm{Map}(X,Y)$ is "uniform convergence on compact sets"; that is to say, $f_1, f_2, \ldots \in \mathrm{Map}(X,Y)$ converges to $f \in \mathrm{Map}(X,Y)$ if $f_1|_K, f_2|_K, \ldots$ converges uniformly to $f|_K$, where $K \subseteq X$ is compact.*

Proof. We know that $f_1, f_2, \ldots$ converges to f iff for every open neighborhood $U \subseteq \mathrm{Map}(X,Y), U \ni f$, all but finitely many $f_i \in U$. WLOG, U is a basic open set w.r.t. the uniform topology; that is,

$$U = \{g \in \mathrm{Map}(X,Y) : g(K_i) \subseteq V_i, \forall i \leq n\}$$

for $K_i \subseteq X$ compact, $V_i \subseteq Y$ open. This condition depends only on $g|_{K_1 \cup \cdots \cup K_n}$. ∎

Note. Given a continuous map

$$p : X' \to X$$

composition with p defines a continuous map

$$\mathrm{Map}(X,Y) \to \mathrm{Map}(X',Y)$$

For example, if $K \subseteq X$ is compact, we get a map $\mathrm{Map}(X,Y) \to \mathrm{Map}(K,Y)$, and $f_1, f_2, \ldots$ converging to f implies $f_1|_K, f_2|_K, \ldots$ converging to $f|_K$ in $\mathrm{Map}(X,Y)$.

Observation 21.2. We pose the question of when Map(X, Y) compact. The answer is essentially never. For instance, consider the sequence of functions x^n, where $X = Y = [0,1]$. These converge pointwise, but they do not converge in Map(X, Y), nor does any subsequence converge.

Definition 21.3. Let X be a topological space, (Y, d) be a metric space. A map $f : X \to Y$ is continuous if

$$(\forall x \in X)(\forall \epsilon > 0)(\exists U \overset{\circ}{\subseteq} X, U \ni x)(\forall x' \in U)$$
$$(d(f(x), f(x')) < \epsilon)$$

Claim 21.4. *f is continuous iff $f^{-1}(B_\epsilon(y))$ is open for all $y \in Y, \forall \epsilon > 0$.*

Proof. Left as exercise. ∎

Definition 21.5. Let X be a topological space, (Y, d) be a metric space. A collection of continuous functions

$$\{f_s : X \to Y\}_{s \in S}$$

is equicontinuous if

$$(\forall x \in X)(\forall \epsilon > 0)(\exists U \overset{\circ}{\subseteq} X, U \ni x)(\forall x' \in U)$$
$$(\forall s \in S)(d(f_s(x), f_s(x')))$$

Example. Let $\{f_s : \mathbb{R} \to \mathbb{R}\}_{s \in S}$ be a collection of functions such that:

1. Each f_s is continuous and differentiable.
2. There exists a uniform bound on the derivatives of each f_s.

Then S is equicontinuous.

Proof. If $|x - x'| < \delta$, then

$$|f_s(x) - f_s(x')| < \delta \cdot \sup_{t \in \mathbb{R}} |f_s(t)| \leq C \cdot \delta$$

It suffices to verify that $|x - x'| < \frac{\epsilon}{C}$. ∎

Theorem 21.6 (Arzela-Ascoli Theorem)**.** *Let X be any locally compact space,*

$$\mathscr{F} = \{f_s \in \mathrm{Map}(X, Y)\}_{s \in S} \subseteq \mathrm{Map}(X, Y)$$

be a family of functions. $\mathscr{F}$ is compact iff

1. *$\forall x \in X, \mathscr{F}_x = \{f_s(x)\} \subseteq Y$ is compact.*
2. *$\mathscr{F}$ is equicontinuous.*
3. *$\mathscr{F}$ is closed in* Map(X, Y).

Example. Let $X = \mathbb{R}, Y = [0,1]$,

$$S = \{f \in [0,1]^{\mathbb{R}} : f \text{ differentiable}, |f'| \leq 1\}$$

(note that we are abusing notation slightly and are using the same symbol S to represent both the collection of functions and its index). S is equicontinuous, and its closure is compact (by Arzela-Ascoli). However, S is not closed.

Note. If f is a uniform limit of differentiable functions, then f need not be differentiable.

Example. Let $X \subseteq \mathbb{C}$ be an open subset, $Y = \{z \in \mathbb{C} : |z| \leq 1\}$. Let

$$S = \{f \in Y^X : f \text{ holomorphic}\}$$

Theorem 21.7 (Cauchy Integral Formula)**.**

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - a} dz$$

If f is holomorphic, then the converse also holds.

Corollary 21.8. *S is closed and equicontinuous. S is also uniformly bounded, since*

$$f'(a) = \frac{1}{-4\pi^2} \oint_C \frac{f(z)}{(z - a)^2} dz$$

By Arzela-Ascoli, S is compact.

Theorem 21.9 (Riemann Mapping Theorem)**.** *Let $U \subsetneq \mathbb{C}$ be simply connected. Then there exists a bijective holomorphic map*

$$f : U \to \{z : |z| < 1\}$$

Proof. (Idea) Let S be defined as before. Take $x \in U$, and let

$$S_0 = \{f \in S : f(x) = 0\} \subset S$$

which is closed and hence compact by Arzela-Ascoli. Consider the function

$$\begin{aligned} S_0 &\longrightarrow \mathbb{R} \\ f &\longmapsto |f'| \end{aligned}$$

We claim that this function is continuous (this is true by the Cauchy Integral Formula). There exists some $f \in S_0$ such that $|f'|$ is as large as possible; we claim that this is our desired function. ∎

Lecture 22 — 10/25/10

Theorem 22.1 (Arzela-Ascoli Theorem)**.** *Let X be a locally compact space, let (Y, d) be a metric space, and let $\mathscr{F} \subseteq \mathrm{Map}(X, Y)$. Then $\overline{\mathscr{F}}$ is compact iff*

1. *$\forall x \in X$, the set $\mathscr{F}_x = \{f(x)\}_{f\in\mathscr{F}}$ has compact closure in Y.*

2. *$\mathscr{F}$ is equicontinuous.*

Proof. Assume $\overline{\mathscr{F}}$ is compact. We want to show that *(1)* and *(2)* are satisfied. WLOG, simply take $\mathscr{F} = \overline{\mathscr{F}}$.

$$\mathscr{F} \subseteq \mathrm{Map}(X, Y) \xrightarrow{\mathrm{ev}_x} Y$$

is continuous since $\mathrm{ev} : X \times \mathrm{Map}(X, Y)$ is continuous. Hence, $\mathrm{ev}_x(\mathscr{F}) \subseteq Y$ is compact, which is condition *(1)*.

It remains to be shown that $\mathscr{F}$ is equicontinuous. Fix $\epsilon > 0$ and take $x \in X$. We know that every map $f \in \mathscr{F}$ is continuous. This means that $\exists U_f \overset{\circ}{\subseteq} X$, $U_f \ni x$ such that $d(f(x), f(x')) < \frac{\epsilon}{2}$ for $x' \in U_f$. By local compactness, assume WLOG that U_f has compact closure. Let

$$V_f = \{g \in \mathrm{Map}(X, Y) : g(\overline{U}_f) \subseteq B_{\frac{\epsilon}{2}}(f(x))\}$$

V_f is open in $\mathrm{Map}(X, Y)$ and $V_f \ni f$. Hence,

$$\mathscr{F} \subseteq \bigcup_{f\in\mathscr{F}} V_f$$

$\mathscr{F}$ is compact, so there are finitely many points $f_1, \dots, f_n \in \mathscr{F}$ such that $\mathscr{F} \subseteq \bigcup_{i\leq n} V_{f_i}$. Let

$$U = \bigcup_{i\leq n} U_{f_i}$$

Note that if $f \in \mathscr{F}$, then also $f \in V_{f_i}$ for some i. Then

$$f(U_{f_i}) \subseteq f(\overline{U}_{f_i}) \subseteq B_{\frac{\epsilon}{2}}(f_i(x))$$

If $x' \in U$, then $\exists i$ such that

$$\begin{aligned} d(f(x'), f(x)) &\leq d(f(x'), f_i(x)) + d(f(x), f_i(x)) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

Hence, $\mathscr{F}$ is equicontinuous.

Now assume that *(1)* and *(2)* hold. We want to show that $\mathscr{F}$ has compact closure in $\mathrm{Map}(X, Y)$. Note that

$$\mathscr{F} \subseteq \mathrm{Map}(X, Y) \subseteq Y^X = \prod_{x\in X} Y$$

Give $\prod_{x\in X} Y$ the product topology, and let $\overline{\mathscr{F}}_\Pi$ be the closure of $\mathscr{F}$ in $\prod_{x\in X} Y$; similarly, we will use ${}_{KU}$ to associate with the compact-open topology. We claim that $\overline{\mathscr{F}}_\Pi$ is equicontinuous (and hence also $\overline{\mathscr{F}}_\Pi \subseteq \mathrm{Map}(X, Y)$).

Fix $\epsilon > 0, x \in X$. We want to find an open neighborhood $U \subseteq X, U \ni x$ such that $d(\bar{f}(x), \bar{f}(x')) < \epsilon$ for all $\bar{f} \in \overline{\mathscr{F}}_\Pi$, $x' \in U$. Since $\mathscr{F}$ is equicontinuous, $\exists U \overset{\circ}{\subseteq} X, U \ni x$ such that $d(f(x), f(x')) < \frac{\epsilon}{3}$ for any $x' \in U, f \in \mathscr{F}$. Note that if $\bar{f} \in \overline{\mathscr{F}}_\Pi$, then there exists $f \in \mathscr{F}$ such that $d(f(x), \bar{f}(x)) < \frac{\epsilon}{3}$ and $d(f(x'), \bar{f}(x')) < \frac{\epsilon}{3}$. So

$$\begin{aligned} d(\bar{f}(x), \bar{f}(x')) &\leq d(\bar{f}(x), f(x)) + d(f(x), f(x')) \\ &\quad + d(f(x'), \bar{f}(x')) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon \end{aligned}$$

Hence it suffices to show that $\overline{\mathscr{F}}_\Pi$ is compact in $\mathrm{Map}_{KU}(X, Y)$. For suppose that this is so. Then $\overline{\mathscr{F}}_\Pi \subseteq \mathrm{Map}_{KU}(X, Y)$ is closed, and hence

$$\overline{\mathscr{F}}_{KU} \subseteq \overline{\mathscr{F}}_\Pi \subseteq \mathrm{Map}_{KU}(X, Y)$$

implies that $\overline{\mathscr{F}}_{KU}$ is compact since it is a closed subset of a compact set, which is our desired conclusion.

First we will show that $\overline{\mathscr{F}}_\Pi \subseteq \prod_{x\in X} Y$ is compact. Recall that $\forall x \in X$, $\mathscr{F}_x \subseteq Y$ has compact closure. Call the closure of this set $K_x \subseteq Y$. Consider

$$\mathscr{F} \subseteq \prod_{x\in X} K_x \subseteq \prod_{x\in X} Y$$

By Tychonoff's Theorem, $\prod_{x\in X} K_x$ is compact, and hence closed. Thus, $\overline{\mathscr{F}}_\Pi \subseteq \prod_{x\in X} K_x$ and hence is also compact.

Finally, we will show that $\mathrm{Map}(X, Y)$ and $\prod_{x\in X} Y$ determine the same topology on $\overline{\mathscr{F}}$. We know that $\mathrm{Map}(X, Y) \to \prod_{x\in X} Y$ continuously, so $\overline{\mathscr{F}}_{KU} \to \overline{\mathscr{F}}_\Pi$ continuously. Thus, any subset $U \subseteq \overline{\mathscr{F}}_\Pi$ open is also open w.r.t. $\mathcal{T}_{KU}$. Now let $U \subseteq \overline{\mathscr{F}}_{KU}$ be open (w.r.t. $\mathcal{T}_{KU}$). We want to show it is also open for the product topology. Let $f \in U$; we will show that $\exists U_f \subseteq U, U_f \ni f$ open w.r.t. $\mathcal{T}_\Pi$.

WLOG, assume U is subbasic open. That is, assume $\exists K \subseteq X$ compact, $W \subseteq Y$ open such that

$$U = \{g \in \overline{\mathscr{F}} : g(K) \subseteq W\}$$

For each $x \in K$, since $f(x) \in W \subseteq Y$, then $\exists \epsilon_x > 0 : B_{\epsilon_x}(f(x)) \subseteq W$. Equicontinuity of $\overline{\mathscr{F}}$ yields $V_x \subseteq X$ open, $V_x \ni x$ such that $d(\bar{f}(x), \bar{f}(x')) < \frac{\epsilon_x}{3}$ for all $\bar{f} \in \overline{\mathscr{F}}, x' \in V_x$. Clearly we have

$$\bigcup_{x\in K} V_x \supseteq K$$

We can choose a finite subcover given by $x_1, \dots, x_n$

$$\bigcup_{i\leq n} V_{x_i} \supseteq K$$

Define

$$U_f = \left\{ g \in \overline{\mathscr{F}} : \forall i \leq n, d(g(x_i), f(x_i)) < \frac{\epsilon_i}{3} \right\}$$

open in $\overline{\mathscr{F}}_\Pi$ and containing f. To complete the proof, it suffices to show $U_f \subseteq U$.

Say $g \in U_f$. Want: $g(K) \subseteq W$. It suffices to show that $g(V_{x_i}) \subseteq W$ for every i. Say $x \in V_{x_i}$. We will show that $g(x) \in B_{\epsilon_{x_i}}(f(x_i)) \subseteq W$. We have that

$$\begin{aligned} d(g(x), f(x_i)) &\leq d(g(x), g(x_i)) + d(g(x_i), f(x_i)) \\ &\quad + d(f(x_i), f(x)) \\ &< \frac{\epsilon_{x_i}}{3} + \frac{\epsilon_{x_i}}{3} + \frac{\epsilon_{x_i}}{3} \\ &= \epsilon \end{aligned}$$

So the compact-open topology and the product topology determine the same topology on $\overline{\mathscr{F}}$, and hence $\overline{\mathscr{F}}$ is compact, as desired. ∎

Lecture 23 — 10/27/10

Recall that a metric space (X, d) is compact iff

1. For every open cover $\mathcal{U}$ of X, $\exists \epsilon > 0 : \forall x \in X$, $B_\epsilon(x) \subseteq U$ for some $U \in \mathcal{U}$.
2. $\forall \epsilon > 0$, there exists a finite covering of X by B_ϵ.

Definition 23.1. Let (X, d) be a metric space. A Cauchy sequence in X is a sequence $x_1, x_2, \ldots \in X$ such that $\forall \epsilon > 0, \exists N \in \mathbb{N} : \forall m, n > N, d(x_m, x_n) < \epsilon$.

Note. Every convergent sequence is a Cauchy sequence. Moreover, in $\mathbb{R}$, the converse holds.

Definition 23.2. A metric space (X, d) is complete if every Cauchy sequence in X is convergent.

Claim 23.3. *Every compact metric space is complete.*

Proof. Let (X, d) be a compact metric space. Say $x_1, x_2, \ldots \in X$ is a Cauchy sequence. Then some subsequence $x_{i_1}, x_{i_2}, \ldots \in X$ converges to x. We can choose N_1 such that $\forall k > N_1, d(x_{i_k}, x) < \frac{\epsilon}{2}$. Moreover, we can choose N_2 such that $\forall m, n > N_2, d(x_n, x_m) < \frac{\epsilon}{2}$. Then $\forall n, i_k > N := \max\{N_1, N_2\}$,

$$d(x_n, x) \leq d(x_n, x_{i_k}) + d(x_{i_k}, x) < \epsilon$$

So $x_1, x_2, \ldots$ converges also to x. ∎

Definition 23.4. Let (X, d) be a metric space. Two Cauchy sequences $x_1, x_2, \ldots \in X$ and $y_1, y_2, \ldots \in X$ are equivalent if $\forall \epsilon > 0, \exists N \in \mathbb{N} : \forall n > N, d(x_n, y_n) < \epsilon$.

Claim 23.5. *This yields an equivalence relation.*

Proof. Easy. ∎

Definition 23.6. The completion of a metric space (X, d) is the set of equivalence classes of Cauchy sequences in X. We will denote the completion as $\overline{X}$. Note that

$$\overline{X} \subseteq X^{\mathbb{N}}$$

Note. There is a natural map $X \to \overline{X}$ given by $x \to x, x, \ldots$.

Claim 23.7. *Let (X, d) be a metric space, $(x_i), (y_i) \in X$ be Cauchy sequences. Define*

$$d'((x_i), (y_i)) = \lim_{n \to \infty} d(x_n, y_n)$$

We claim that this limit exists.

Proof. It suffices to show that $\forall \epsilon > 0$, $\exists N : \forall n, m > N$,

$$|d(x_n, y_n) - d(x_m, y_m)| < \epsilon$$

We can choose N_1 such that $\forall n, m, > N_1, d(x_n, x_m) < \frac{\epsilon}{2}$; we can choose N_2 such that $\forall n, m > N_2, d(y_n, y_m) < \frac{\epsilon}{2}$. Take $N = \max\{N_1, N_2\}$. Then $\forall n, m > N$,

$$\begin{aligned} |d(x_n, y_n) - d(x_m, y_m)| &\leq |d(x_n, y_n) - d(x_m, y_n)| \\ &\quad + |d(x_m, y_n) - d(x_m, y_m)| \\ &\leq d(x_n, x_m) + d(y_n, y_m) \\ &< \epsilon \end{aligned}$$

∎

Definition 23.8. We make $\overline{X}$ into a metric space with the following metric

$$\bar{d}(\bar{x}, \bar{y}) = d'((x_i), (y_i))$$

for any $(x_i) \in \bar{x}, (y_i) \in \bar{y}$.

Claim 23.9. *$\bar{d}$ is a metric.*

Proof. We note that $d'((x_i), (y_i)) = 0 \iff (x_i) \sim (y_i)$. The proof is easy. ∎

Note. Note that our canonical map $X \to \overline{X}$ is an isometry

$$d(x, y) = d'((x), (y)) = \bar{d}(\bar{x}, \bar{y})$$

Claim 23.10. *$\overline{X}$ is complete.*

Proof. Choose a Cauchy sequence $\bar{x}_1, \bar{x}_2, \bar{x}_3, \ldots \in \overline{X}$. WLOG, we can represent each $\bar{x}_i$ by a Cauchy sequence $(x_{i,j})_j$ such that

$$d(x_{i,j}, x_{i,j'}) < \frac{1}{n}, \qquad \forall j, j' > n$$

We claim that $\bar{x}_1, \bar{x}_2, \ldots$ converges to the equivalence class of the Cauchy sequence $x_{1,1}, x_{2,2}, \ldots$, which we will denote $\bar{x}$. First, we will show that $x_{i,i}$ is a Cauchy sequence. Fix $\epsilon > 0$. Choose N such that $\forall n, m > N$, $\bar{d}(\bar{x}_n, \bar{x}_m) < \frac{\epsilon}{2}$. Then $\forall n, m > N$,

$$\begin{aligned} d(x_{n,n}, x_{m,m}) &\le d(x_{n,n}, \bar{x}_n) + \bar{d}(\bar{x}_n, \bar{x}_m) + d(\bar{x}_m, x_{m,m}) \\ &< \frac{1}{n} + \frac{\epsilon}{2} + \frac{1}{m} \\ &< \frac{\epsilon}{2} + \frac{2}{N} \end{aligned}$$

Note that this is an abuse of notation; when we write $\bar{x}_n$ and $\bar{x}_m$, we actually mean some term sufficiently far along in their representative sequences. WLOG, take N such that $\frac{2}{N} < \frac{\epsilon}{2}$. Then $d(x_{n,n}, x_{m,m}) < \epsilon$.

Now we must show that $\bar{x}_1, \bar{x}_2, \ldots$ converges to $\bar{x}$. Fix $\epsilon > 0$. We can choose N_1 such that $\forall n > N_1$, $d(\bar{x}_n, x_{n,n}) \le \frac{1}{n}$. We can also choose N_2 such that $\forall n, m > N_2$, $d(x_{n,n}, x_{m,m}) < \frac{\epsilon}{2}$. Take $N = \max\{N_1, N_2\}$ and $n, m > N$. Then

$$\bar{d}(\bar{x}_n, \bar{x}) \le d(\bar{x}_n, x_{n,n}) + d(x_{n,n}, x_{m,m}) < \frac{1}{n} + \frac{\epsilon}{2}$$

■

Enlarging N gives us our desired conclusion.

Lecture 24 — 10/29/10

Claim 24.1. *If X is complete and $Y \subseteq X$, then Y is closed iff Y is complete.*

Proof. Recall that Y is closed iff $\forall y_1, y_2, \ldots \in Y$ converging to $x \in X$, we have $y \in Y$. If Y is complete, then $\{y_i\}$ is Cauchy and therefore converges to $y \in Y$. So $x = y \in Y$. Conversely, if Y is closed and $\{y_i\}$ is a Cauchy sequence in Y, then $y_i \to x \in X$. If Y is closed, $x \in Y$ and therefore $\{y_i\}$ converges in Y. ■

Definition 24.2. Let (X, d) be a metric space. We say X is totally bounded if $\forall \epsilon > 0$, there exists a finite covering of X by ϵ-balls.

Theorem 24.3 (Heine-Borel Theorem)**.** *Let (X, d) be a metric space. X is compact iff X is complete and totally bounded.*

Proof. We have previously proven the only if direction (Theorem 10.5), so we will concern ourselves only with the if direction. Assume that X is complete and totally bounded. We want to show that every sequence $x_1, x_2, \ldots X$ has a convergent subsequence.

Fix $\epsilon > 0$. We will construct a subsequence $y_1, y_2, \ldots$ of (x_i) such that $d(y_i, y_j) < \epsilon, \forall i, j$. Choose a covering of X by finitely many $\frac{\epsilon}{2}$-balls, $B_1, \ldots, B_m$. Each $x_i \in B_k$ for some k. Then one of the B_k must contain infinitely many of the x_i, which yields our desired subsequence.

We can thusly construct a collection of subsequences for $\epsilon = 1, \frac{1}{2}, \frac{1}{3}, \ldots$

$$\begin{gathered} x_{1,1}, x_{1,2}, x_{1,3}, \ldots \\ x_{2,1}, x_{2,2}, x_{2,3}, \ldots \\ \vdots \end{gathered}$$

such that each $(x_{n,i})$ contains $(x_{n+1,i})$ as a subsequence, and such that $d(x_{n,i}, x_{n,j}) < \frac{1}{n}, \forall i, j$. Consider the sequence $x_{1,1}, x_{2,2}, \ldots \in X$. This is a subsequence of (x_i). If $i < j$, then

$$d(x_{i,i}, x_{j,j}) = d(x_{i,i}, x_{i,k}) < \frac{1}{i}$$

for some k, since the j-sequence is a subsequence of the i-sequence. So $(x_{i,i})$ is Cauchy and hence convergent. Thus, every sequence has a convergent subsequence, so X is compact. ■

Remark. Given a metric space (X, d), how can we characterize the completion $\overline{X}$? We might guess that $\overline{X}$ is universal among complete metric spaces Y with $X \hookrightarrow Y$. That is, we might expect that, given a continuous map $f : X \to Y$, f extends uniquely to a map $\bar{f} : \overline{X} \to Y$.

However, this is not the case. The function $x \mapsto \frac{1}{x-\pi}$ is a continuous map $\mathbb{Q} \to \mathbb{R}$, and it extends uniquely to a map $\overline{\mathbb{Q}} \cong \mathbb{R} \to \mathbb{R}$, but this extension is not continuous.

We want to be able to choose $\overline{x} \in \overline{X}$ and represent it by a Cauchy sequence (x_i). Consider $f(x_1), f(x_2), \ldots$. If this converges to a point y, we may set $\bar{f}(\overline{x}) = y$. But as in the previous example, the sequence $f(x_1), f(x_2), \ldots$ may not be Cauchy, or it may depend on the choice of representative $\{x_i\}$.

Definition 24.4. A function $f : X \to Y$ between metric spaces $(X, d_X), (Y, d_Y)$ is uniformly continuous if

$$\begin{aligned} &(\forall \epsilon > 0)(\exists \delta > 0)(\forall x, x' \in X) \\ &\qquad (d_X(x, x') < \delta \Longrightarrow d_Y(f(x), f(x')) < \epsilon) \end{aligned}$$

Proposition 24.5. *Let $f : X \to Y$ be a uniformly continuous map between metric spaces $(X, d_X), (Y, d_Y)$, where Y is complete. Then f extends uniquely to a map $\bar{f} : \overline{X} \to Y$.*

Proof. We will abuse notation and denote all metrics d; their meanings should be clear in context. Let $x_1, x_2, \ldots$ be a Cauchy sequence. We want to show $f(x_i)$'s are Cauchy. Fix $\epsilon > 0$. We want to find $n \in \mathbb{N}$: $d(f(x_i), f(x_j)) < \epsilon$ for $i, j > n$. By uniform continuity, $\exists \delta > 0 : d(x, x') < \delta \Longrightarrow d(f(x), f(x')) < \epsilon$. We need $n \in \mathbb{N}$ such that $d(x_i, x_j) < \delta$ for $i, j > n$. But this is true for $n \gg 0$ since (x_i) is Cauchy.

Choose $\overline{x} \in \overline{X}$ and represent it by a Cauchy sequence $(x_i) \in X$. Then $f(x_i) \in Y$ is Cauchy and hence convergent to some point $y \in Y$ since Y is complete. Define $\bar{f} : \overline{X} \to Y$ to extend f by

$$\bar{f}(\overline{x}) = y$$

First, we must show that $\bar{f}$ is well-defined. Suppose that $(x_i), (x'_i)$ are both representatives of $\overline{x}$. Fix $\epsilon > 0$. We want $n \in \mathbb{N} : \forall i > n, d(f(x_i), f(x'_i)) < \epsilon$. It suffices to show that $\forall i > n, d(x_i, x'_i) < \delta$. This is true since $(x_i) \sim (x'_i)$. Hence, $f(x_i) \sim f(x'_i)$, as desired.

Finally, we must show that $\bar{f}$ is uniformly continuous. Take $\overline{x}, \overline{x}' \in \overline{X}$, and let $(x_i), (x'_i)$ be respective representatives. Fix $\epsilon > 0$. Then there is $\delta > 0$ such that $d(x_i, x'_i) < \delta \implies d(f(x_i), f(x'_i)) = d(\bar{f}(x_i), \bar{f}(x'_i)) < \frac{\epsilon}{2}$. If $d(\overline{x}, \overline{x}') < \delta$, there is some $n \gg 0$ such that $\forall i > n$, $d(x_i, x'_i) < \delta$. Thus, taking limits

$$d(\bar{f}(\overline{x}), \bar{f}(\overline{x}')) \leq \frac{\epsilon}{2} < \epsilon$$

So $\bar{f}$ is uniformly continuous. ∎

Proposition 24.6. *Let X be a metric space, let $i : X \to X'$ be a uniformly continuous map, where X' a complete metric space such that $\forall Y$ complete, $X \hookrightarrow Y$, every uniformly continuous function extends uniquely as*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i"'] & Y \\ X' \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

Then $X' \cong \overline{X}$.

Lecture 25 — 11/1/10

We want to formalize the following notion:

1. Consider polynomials $f(x) = a_0 + a_1 x + \cdots + a_n x^n$. "Generally, the roots of f are distinct."
2. Consider two lines, $L = \{(x, y) : ax + by = c\}$ and $L' = \{(x, y) : a'x + b'x = c'\}$. "Generally, L and L' intersect in one point."

Remark. One way to do this is measure theory. Assign to each set S a "measure" $\mu(S)$ such that $\mu(S) = 0$ if S is in some sense "small," and where $\mu(S \sqcup S') = \mu(S) + \mu(S')$.

Remark. We can also take a topological approach. We observe that, in general, open sets (that are nonempty) have many points. We want additionally to demand some property stronger than being nonempty, such that intersecting "large" sets preserves "largeness." Recall that if X is a topological space, a subset $S \subseteq X$ is dense if $\overline{S} = X$. A better formulation of our idea is that dense open sets are large.

Claim 25.1. *If $U, V \subseteq X$ are dense and open, then $U \cap V$ is dense and open.*

Proof. $U \cap V$ is certainly open. Let W be a nonempty open set in X. $U \cap W$ is nonempty and open. Then $U \cap W \cap V \neq \emptyset$, or in other words, $(U \cap V) \cap W \neq \emptyset$, as desired. ∎

Remark. The condition of containing a dense open set, however, is too restrictive. The intersection of a countable collection of dense open sets need not be open, though it often is dense.

Definition 25.2. A topological space X is a Baire space if any countable intersection of dense open sets is dense.

Theorem 25.3 (Baire Category Theorem)**.**

1. *Any locally compact Hausdorff space is Baire.*
2. *Any complete metric space is Baire.*

Proof. Let X be locally compact Hausdorff. Let $U_1, U_2, \ldots \subseteq X$ be dense open sets. Let $V \subseteq X$ be nonempty. We want to show that $V \cap U_1 \cap U_2 \cap \cdots \neq \emptyset$. We know $U_1 \cap V \neq \emptyset$. Then there exists a nonempty open set V_1 such that $\overline{V}_1$ is compact and $\overline{V}_1 \subseteq U_1 \cap V$. $V_1 \cap U_2 \neq \emptyset$, so there is a nonempty open set V_2 such that $\overline{V}_2 \subseteq V_1 \cap U_2$ is compact. We get a sequence of sets

$$\begin{gathered} V \supseteq \overline{V}_1 \subseteq U_1 \\ V_1 \supseteq \overline{V}_2 \subseteq U_2 \\ V_2 \supseteq \overline{V}_3 \subseteq U_3 \\ \vdots \end{gathered}$$

$\bigcap \overline{V}_i \neq \emptyset$ by compactness of $\overline{V}_1$, so we can choose $x \in \bigcap \overline{V}_i$. Then $x \in V, x \in U_i$ for all i.

Now assume that (X, d) is a complete metric space. Let U_i, V be as above. Then $U_1 \cap V \neq \emptyset$, so $\exists x_1 \in U_1 \cap V$. There is $\epsilon_1 > 0$ such that $B_{2\epsilon_1}(x_1) \subseteq U_1 \cap V$; WLOG, $\epsilon_1 \leq 1$. Define $V_1 = B_{\epsilon_1}(x_1)$. $V_1 \cap U_2 \neq \emptyset$, so $\exists x_2 \in V_1 \cap U_2$, $\exists \epsilon_2 > 0, \epsilon_2 < \frac{1}{2} : B_{2\epsilon_2}(x_2) \subseteq V_1 \cap V_2$. Define $V_2 = B_{\epsilon_2}(x_2)$. We get a sequence of points $x_1, x_2, \ldots$ and positive real numbers $0 < \epsilon_i < \frac{1}{i}$ such that, defining $V_i = B_{\epsilon_i}(x_i)$, we have

$$V \supseteq V_1 \supseteq V_2 \supseteq \cdots, \quad V_i \subseteq U_i \implies \overline{V}_i \subseteq U_i$$

The sequence $x_1, x_2, \ldots$ is a Cauchy sequence, since $\forall i \leq j$, $x_j \in V_j \subseteq V_i = B_{\epsilon_i}(x_i)$, so $d(x_i, x_j) < \epsilon_i < \frac{1}{i}$. By completeness, the sequence $x_1, x_2, \ldots$ converges to some $x \in X$. We have

$$d(x_i, x) = \lim_{j \to \infty} d(x_i, x_j) \leq \epsilon_i < 2\epsilon_i$$

Then $x \in U_i$ for every i, and $x \in V$. ∎

Definition 25.4. Let X be a topological space (usually Baire). A set $S \subseteq X$ is generic if there exist dense open sets $U_1, U_2, \dots$ such that $S \supseteq \bigcap U_i$.

Corollary 25.5. *If X is Baire, any generic subset is dense.*

Definition 25.6. Let X be a topological space (usually Baire). A set $S \subseteq X$ is meagre if $X - S$ is generic.

Definition 25.7. Let X be a topological space. A set $A \subseteq X$ is nowhere dense if $A \cap V$ is not dense in V for any $V \overset{\circ}{\subseteq} X$.

Note. Let X be a Baire space. An open set $U \subseteq X$ is dense and open iff $X - U$ is closed and nowhere dense; that is, $(X - U) \cap V$ is not dense in V for any V. A meagre set, therefore, is a set $S \subseteq \bigcup Y_i$ where each Y_i is closed and nowhere dense.

Note. A countable union of meagre sets is meagre, and a countable intersection of generic sets is generic.

Example. There exists a decomposition $\mathbb{R} = A \cup B$ where A has Lebesgue measure zero and B is meagre. These two notions of "smallness" are incompatible.

Lecture 26 — 11/3/10

Recall that a continuous bijection $f : X \to Y$ between topological spaces need not be a homeomorphism; that is, f^{-1} need not be continuous. Equivalently, for any $U \subseteq X$ open, $f(U)$ need not be open, and for any $K \subseteq X$ closed, $f(K)$ need not be closed.

Definition 26.1. Let $f : X \to Y$ be a continuous map between topological spaces. f is open if $\forall U \subseteq X$ open, $f(U) \subseteq Y$ is open. Similarly, f is closed if $\forall K \subseteq X$ closed, $f(K) \subseteq Y$ is closed.

Note. If $f : X \to Y$ is a continuous bijection, TFAE:

1. f is a homeomorphism.
2. f is open.
3. f is closed.

Example. Let X be a compact topological space, Y a Hausdorff space. Then any continuous map $f : X \to Y$ is closed.

Proof. If $K \subseteq X$ is closed, K is compact. Then $f(K)$ is compact in Y, and hence closed. ■

Corollary 26.2. *Let $f : X \to Y$ be a continuous bijection between topological spaces. If X is compact and Y is Hausdorff, f is a homeomorphism.*

Example. Let X, Y be any topological spaces. Then the projection maps $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are open.

Proof. Let $U \overset{\circ}{\subseteq} X \times Y$. Since $\pi_X(\bigcup U_\alpha) = \bigcup \pi_X(U_\alpha)$, assume WLOG that U is basic open. Then $U = V \times W : V \overset{\circ}{\subseteq} X$ open and $W \overset{\circ}{\subseteq} Y$ open. Hence,

$$\pi_X(U) = \begin{cases} \emptyset & W = \emptyset \\ V & W \neq \emptyset \end{cases}$$

and thus is open. By symmetry, π_Y is also open. ■

Note. Projection maps are generally not closed. Let $X = Y = \mathbb{R}$, and let $K = \{(x, y) : xy = 1\} \subseteq \mathbb{R}^2$. Then $\pi_X(K) = \mathbb{R} - \{0\}$ which is not closed.

Proposition 26.3. *Let X, Y be any topological spaces, Y compact. Then the projection map $\pi_X : X \times Y \to X$ is closed.*

Proof. Let $K \subseteq X \times Y$ be closed. We want to show that $\pi_X(K)$ is closed, or equivalently that $U := X - \pi_X(K)$ is open. Then

$$U = \{x \in X : \forall y \in Y, (x, y) \notin K\}$$

Say $x \in U$. Then $\forall y \in Y, (x, y) \notin K$. Since K is closed, WLOG, for each y, there is a basic neighborhood $V_y \times W_y$ given by $V_y \overset{\circ}{\subseteq} X, V_y \ni x$ and $W_y \overset{\circ}{\subseteq} Y, W_y \ni y$ such that $V_y \times W_y \cap K = \emptyset$. The sets $\{W_y\}$ cover Y and hence admit a finite subcover $W_{y_i}, W_{y_2}, \dots, W_{y_n}$. Let $V = V_{y_i} \cap \cdots \cap V_{y_n}$. Clearly $x \in V \overset{\circ}{\subseteq} U$. If $x' \in V$, since every $y \in Y$ is in some W_{y_i}, then $(x', y) \in V_{y_i} \times W_{y_i}$, and hence $(x', y) \notin K$. This shows that U is open, and hence $\pi_X(K)$ is closed, as desired. ■

Corollary 26.4. *There exists a function $f : [0, 1] \to \mathbb{R}$ that is continuous but nowhere differentiable. Moreover, the set*

$$S = \left\{ f \in \operatorname{Map}([0, 1], \mathbb{R}) : \exists x \in [0, 1), \tfrac{\mathrm{d}f}{\mathrm{d}t} \textit{ is defined at } x \right\}$$

is meagre.

Proof. For each n, define $E_n \subseteq \operatorname{Map}([0, 1], \mathbb{R})$ by

$$E_n = \Big\{ f \in \operatorname{Map}([0,1], \mathbb{R}) : \big(\exists x \in [0, 1 - \frac{1}{n}]\big) \\ \big(\forall \epsilon < \frac{1}{n}\big)\big(|f(x + \epsilon) - f(x)| \leq n\epsilon\big) \Big\}$$

We will make three claims:

1. If f is differentiable at any point $x \in [0, 1]$, then $f \in E_n$ for some n.
2. Each E_n is closed.
3. Each E_n is nowhere dense.

These imply that the collection S of somewhere-differentiable functions is contained in $\bigcup E_n$, which is a countable union of closed, nowhere dense sets, and hence S is meagre in $\mathrm{Map}([0,1],\mathbb{R})$ (since it is complete).

1. Suppose $\frac{\mathrm{d}f}{\mathrm{d}t}$ exists at $t=x$ and takes the value c. Then $\exists\epsilon>0 : |f(x+\epsilon)-f(x)| \le 2|c|\epsilon$ for small enough ϵ, and hence for $\epsilon<\frac{1}{n}$ for some n. Let $n' \ge \frac{n}{2|c|}$. Then $f \in E_{n'}$.

2. Define $K_n \subset \mathrm{Map}([0,1],\mathbb{R}) \times [0,\frac{1}{n}] \times [0,1-\frac{1}{n}]$ by
$$K_n = \{(f,\epsilon,x) : |f(x+\epsilon)-f(x)| \le n\epsilon\}$$
Note that K_n is closed. $f \in E_n$ iff $\exists x \in [0,1-\frac{1}{n}]$ such that $\forall\epsilon \in [0,\frac{1}{n}]$, $(f,x,\epsilon) \in K_n$. Let us define $A_n \subseteq \mathrm{Map}([0,1],\mathbb{R}) \times [0,1-\frac{1}{n}]$ by
$$A_n = \{(f,x) : \forall\epsilon \in [0,\frac{1}{n}] : (f,\epsilon,x) \in K_n\}$$
The complement of A_n is the image of the complement of K_n under the relevant projection map. The projection map is open, and hence A_n is closed. E_n is the image of A_n under closed projection (the projection map is closed since $[0,1-\frac{1}{n}]$ is compact).

3. We claim it suffices to show that E_n does not contain any nonempty open set U. For suppose this is the case, and suppose that E_n is not also nowhere dense. Then $\exists V \overset{\circ}{\subseteq} \mathrm{Map}([0,1],\mathbb{R})$ such that $E_n \cap V$ is dense in V. But since E_n is closed, $E_n \cap V$ is dense in V iff $E_n \cap V = V \iff V \subseteq E_n$. $\Rightarrow\Leftarrow$.

   Suppose $\exists U \overset{\circ}{\subseteq} E_n$. Choose $f \in U$. Then $\exists\epsilon>0 : B_\epsilon(f) \subseteq U$. Since f is continuous and $[0,1]$ is compact, f is also uniformly continuous. Hence, $\exists\delta>0 : |x-x'|<\delta \Longrightarrow |f(x)-f(x')|<\frac{\epsilon}{3}$. WLOG, $\delta=\frac{1}{N}$ for some $N\in\mathbb{N}$. Let g be a piecewise linear function $[0,1]\to\mathbb{R}$ which agrees with f on $\frac{i}{N}$ for $0\le i\le N$. If $x\in[0,1]$, $x\in[\frac{i}{N},\frac{i+1}{N}]$ for some i. Then
$$\begin{aligned}|g(x)-f(x)| &\le \left|g(x)-f\big(\tfrac{i}{N}\big)\right| + \left|f\big(\tfrac{i}{N}\big)-f(x)\right| \\ &\le \left|f\big(\tfrac{i+1}{N}\big)-f\big(\tfrac{i}{N}\big)\right| + \frac{\epsilon}{3} \\ &\le \frac{2\epsilon}{3} \\ &< \epsilon\end{aligned}$$
   Hence, $B_\epsilon(f) \supseteq B_{\epsilon/3}(g)$. Let
$$f_N(x) := g(x) + \frac{1}{N}\sin(N^2x)$$
   For $N \gg 0$, we know that $f_N(x) \in B_{\epsilon/3}(g)$. But for very large N, we must have $f_N(x) \notin E_n$. So E_n is nowhere dense.

By our above argument, this completes the proof. ■

Lecture 27 — 11/5/10

Consider the problem of trying to define $\log : \mathbb{C} \xrightarrow{?} \mathbb{C}$. This is not possible, even as a map from $\mathbb{C}^* \to \mathbb{C}$ (where $\mathbb{C}^* = \mathbb{C} - \{0\}$). What properties can we try to preserve?

1. $\dfrac{\partial \log(z)}{\partial z} = \dfrac{1}{z}$.
2. $\log(1) = 0$.

We want to show, roughly, that these properties determine $\log z$ up to a constant. Specifically, taking any $z \in \mathbb{C}$, choose a smooth curve C between z and 1 not passing through 0, and define
$$\log z = \int_C \frac{1}{z}$$
We want to eliminate ambiguity by choosing C to be a line, but this is not always possible (since it might pass through 0). Instead consider the semicircle $\theta \mapsto e^{i\theta}$. If $z = e^{i\theta}$, this gives $\log z = i\theta$, which yields $\log(-1) = i\pi$. But if we instead take $\theta \mapsto e^{-i\theta}$, we get $\log(-1) = -i\pi$. The problem is that there is a "hole" in $\mathbb{C}^*$, and there are different ways to "go around" this hole. We begin our foray into algebraic topology by trying to formalize this problem.

Definition 27.1. Let X be a topological space. Recall that a path in X is a continuous function $p : [0,1] \to X$. We say p is a path from $p(0) \in X$ to $p(1) \in X$.

Recall also that X is path-connected if $\forall x,y \in X$, $\exists p : [0,1] \to X$ a path such that $p(0)=x$, $p(1)=y$. More generally, define $\sim$ on X such that $x \sim y$ if there exists a path p from x to y. We say that $x,y \in X$ are in the same path component if $x \sim y$.

Claim 27.2. *$\sim$ is an equivalence relation.*

Proof.

1. $x \sim x$. We can choose the constant function taking $[0,1] \to \{x\}$.
2. $x \sim y \Longrightarrow y \sim x$. If $p : [0,1] \to X$ satisfies $p(0)=x$, $p(1)=y$, set $q(t)=p(1-t)$, which satisfies $q(0)=y$, $q(1)=x$.
3. $x \sim y, y \sim z \Longrightarrow x \sim z$. Let p be a path between x and y, and q a path between y and z. Define
$$r(t) = \begin{cases} p(2t) & 0 \le t \le \frac{1}{2} \\ q(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

This shows that a path component in X is an equivalence class under $\sim$. ■

Note.

1. Any topological space X is the union of its path-components, each of which is path-connected.

2. If $x \sim y$, $\exists p : [0,1] \to X$, $p(0) = x$, $p(1) = y$. Then $\forall t \in [0,1]$, $p(t) \sim x$. Hence, p is a map $[0,1] \to \{a \in X : a \sim x\}$.

3. The topology on X cannot be recovered from the topology of path components (c.f. disjoint clopen sets, disconnected subspaces, topologist's sine curve), but often can be in practice.

Definition 27.3. Let X be a topological space, $x, y \in X$. The path space $P_{x,y}(X)$ is defined by

$$P_{x,y}(X) = \{p \in \mathrm{Map}([0,1], X) : p(0) = x, p(1) = y\}$$

$P_{x,y}(X)$ inherits the topology on $\mathrm{Map}([0,1], X)$. Similarly, the space $P_{x,x}(X)$ is called the loop space at x.

Definition 27.4. Two paths $p, q : [0,1] \to X$ with $p(0) = q(0) = x$, $p(1) = q(1) = y$ are homotopic if p, q lie in the same path component of $P_{x,y}(X)$. Equivalently, $\exists$ a path

$$\lambda : [0,1] \to P_{x,y}(X)$$

with $\lambda(0) = p$, $\lambda(1) = q$. λ gives the same data as a continuous map

$$h : [0,1] \times [0,1] \to X$$

For each $t \in [0,1]$, $\lambda(t)$ is a path $t \mapsto h(s,t)$

- $\lambda(0) = p$ means $h(0,t) = p(t)$ for $s \in [0,1]$.
- $\lambda(1) = q$ means $h(1,t) = q(t)$ for $s \in [0,1]$.
- $h(s,0) = x$ for all $s \in [0,1]$.
- $h(s,1) = y$ for all $s \in [0,1]$.

We say that h is a homotopy from p to q.

Definition 27.5. Let X be any topological space. Define

$$\pi_0 X = X/\sim$$

to be the set of path components. Homotopy is likewise an equivalence relation, and we call the equivalence classes under homotopy the homotopy classes, denoted $[p] \in \pi_0 P_{x,y}(X)$ for $p \in P_{x,y}(X)$.

Definition 27.6. Let X be a topological space. The fundamental group of X at the base point $x \in X$ is

$$\pi_1(X, x) = \pi_0 P_{x,x}(X)$$

the set of loops starting and ending at x, modulo homotopy.

Observation 27.7. Let $f : X \to Y$ be a continuous map of topological spaces. If $x, x' \in X : x \sim x'$, then $f(x) \sim f(x')$. Thus, f induces a map

$$\pi_0 X \to \pi_0 Y$$

Note. The fundamental group $\pi_1(X, x)$ depends on the choice of base point $x \in X$, but this dependence is minimal if X is path connected. Suppose so, and say $p : [0,1] \to X$ is a path, $p(0) = x$, $p(1) = y$. This gives a map

$$P_{x,x}(X) \to P_{y,y}(X)$$

that takes $\lambda \in P_{x,x}(X)$ to $\lambda' \in P_{y,y}(X)$, where λ' is defined by

$$\lambda'(t) = \begin{cases} p(1-3t) & 0 \le t \le \frac{1}{3} \\ \lambda(3t-1) & \frac{1}{3} \le t \le \frac{2}{3} \\ p(3t-2) & \frac{2}{3} \le t \le 1 \end{cases}$$

It is left as an exercise to show that $\lambda \mapsto \lambda'$ is continuous. This gives a map

$$\pi_1(X, x) = \pi_0 P_{x,x}(X) \longrightarrow \pi_0 P_{y,y}(X) = \pi_1(X, y)$$

The maps $\pi_1(X, x) \to \pi_1(X, y)$ and $\pi_1(X, y) \to \pi_1(X, x)$ induced by p and p^{-1} are themselves inverses. See Proposition 28.6.

Lecture 28 — 11/8/10

Recall that if X is a topological space, and $x, y \in X$,

$$P_{x,y} = \{p \in \mathrm{Map}([0,1], X) : p(0) = x, p(1) = y\}$$

is the collection of paths from x to y, and is a topological space w.r.t. the compact-open topology. Recall also that

$$\pi_0 P_{x,y} = P_{x,y}/\sim$$

denotes the set of homotopy classes of paths from x to y.

Claim 28.1. *Consider the topological space $\mathbb{C}^*$. Then the paths from 1 to -1 given by the upper and lower semicircles are not homotopic.*

Definition 28.2. Let $p, q : [0,1] \to X$ be paths such that $p(1) = q(0)$. Define the product $p * q : [0,1] \to X$ by

$$(p * q)(t) = \begin{cases} p(2t) & 0 \le t \le \frac{1}{2} \\ q(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

Note that this is just the concatenation of p and q, taking both paths at double speed.

Claim 28.3. *$p * q$ is continuous.*

Proof. Left as exercise. ∎

Claim 28.4. *By the above definition, we get a map*

$$\mu : P_{x,y} \times P_{y,z} \to P_{x,z}$$

μ is continuous.

Proof. Left as exercise. ■

Remark. μ induces a map

$$\pi_0(P_{x,y} \times P_{y,z}) \cong \pi_0(P_{x,y}) \times \pi_0(P_{y,z}) \longrightarrow \pi_0(P_{x,z})$$

That is, if $p \sim p', q \sim q'$, then $p * q \sim p' * q'$. Hence, $*$ is defined also for homotopy classes, by

$$[p] * [q] = [p * q]$$

Proposition 28.5. *$\pi_1(X, x)$, along with multiplication μ, is a group.*

Proof. Define the path 1 by

$$\begin{aligned} 1 : [0,1] &\longrightarrow X \\ t &\longmapsto x \end{aligned}$$

We claim that 1 is a left and right identity in $\pi_1(X, x)$. To show that 1 is a right identity, we need to show that $\forall p \in P_{x,x}$, $p * 1 \sim p$.

$$(p * 1)(t) = \begin{cases} p(2t) & 0 \le t \le \frac{1}{2} \\ x & \frac{1}{2} \le t \le 1 \end{cases}$$

We define a map $h : [0,1] \times [0,1] \to X$ by

$$h(s,t) = \begin{cases} p((2-s)t) & 0 \le t \le \frac{1}{2-s} \\ x & \frac{1}{2-s} \le t \le 1 \end{cases}$$

We have $h(0,t) = (p * 1)(t)$ and $h(1,t) = p(t)$, and also $h(s,0) = h(s,1) = x$, so h is a homotopy from $p * 1$ to p. The proof that 1 is a left identity is similar.

Now we claim that every element of $\pi_1(X, x)$ has a two-sided inverse, and in particular, this inverse is given by the path

$$\bar{p}(t) = p(1-t)$$

We want to show that $p * \bar{p}$ and $\bar{p} * p$ are homotopic to 1.

$$(p * \bar{p})(t) = \begin{cases} p(2t) & 0 \le t \le \frac{1}{2} \\ p(2-2t) & \frac{1}{2} \le t \le 1 \end{cases}$$

We make a map $h : [0,1] \times [0,1] \to X$

$$h(s,t) = \begin{cases} x & 0 \le t \le \frac{s}{2} \\ p(2t-s) & \frac{s}{2} \le t \le \frac{1}{2} \\ p(2-2t-s) & \frac{1}{2} \le t \le 1 - \frac{s}{2} \\ x & 1 - \frac{s}{2} \le t \le 1 \end{cases}$$

Note that $h(0,t) = (p * \bar{p})(t)$ and $h(1,t) = 1(t)$, and also $h(s,0) = h(s,1) = x$, so h is a homotopy from $p * \bar{p}$ to 1. The proof for $\bar{p} * p$ is similar.

Finally, we want to show that μ is associative. We claim that $(p*q)*r \sim p*(q*r)$ for all $p \in P_{w,x}$, $q \in P_{x,y}$, $r \in P_{y,z}$.

$$(p * q) * r = \begin{cases} p(4t) & 0 \le t \le \frac{1}{4} \\ q(4t-1) & \frac{1}{4} \le t \le \frac{1}{2} \\ r(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

whereas

$$p * (q * r) = \begin{cases} p(2t) & 0 \le t \le \frac{1}{2} \\ q(4t-2) & \frac{1}{2} \le t \le \frac{3}{4} \\ r(4t-3) & \frac{3}{4} \le t \le 1 \end{cases}$$

We make a map $h : [0,1] \times [0,1] \to X$

$$h(s,t) = \begin{cases} r((4-2s)t) & 0 \le t \le \frac{1}{4-2s} \\ q(4(t - \frac{1}{4-2s})) & \frac{1}{4-2s} \le t \le \frac{1}{4} + \frac{1}{4-2s} \\ p((2+2s)(t - \frac{1}{4} - \frac{1}{4-2s})) & \frac{1}{4} + \frac{1}{4-2s} \le t \le 1 \end{cases}$$

We have $h(0,t) = ((p*q)*r)(t)$ and $h(1,t) = (p*(q*r))(t)$, and also $h(s,0) = w$, $h(s,1) = z$. Hence, h is a homotopy, which completes our proof. ■

Proposition 28.6. *Let $p \in P_{x,y}(X)$. The map*

$$\begin{aligned} \varphi_p : \pi_1(X,x) &\longrightarrow \pi_1(X,y) \\ [q] &\longmapsto [\bar{p}] * [q] * [p] \end{aligned}$$

determines an isomorphism of groups.

Proof. First we claim that φ_p is a homomorphism of groups; that is, $\varphi_p([q] * [q']) = \varphi_p([q]) \circ \varphi_p([q'])$.

$$\begin{aligned} \varphi_p([q] * [q']) &= [\bar{p}] * [q] * [q'] * [p] \\ &= ([\bar{p}] * [q] * [p]) * ([\bar{p}] * [q'] * [p]) \\ &= \varphi_p([q]) \circ \varphi_p([q']) \end{aligned}$$

Then we claim that φ_p is bijective. It has an inverse

$$\varphi_{\bar{p}} : \pi_1(X,y) \to \pi_1(X,x)$$

For $[q] \in \pi_1(X,x)$,

$$\begin{aligned} \varphi_{\bar{p}}(\varphi_p([q])) &= \varphi_{\bar{p}}([\bar{p}] * [q] * [p]) \\ &= [p] * [\bar{p}] * [q] * [p] * [\bar{p}] \\ &= [q] \end{aligned}$$

This completes the proof. ■

Note. If X is path-connected, this means the choice of base point for the fundamental group is irrelevant, up to isomorphism. However, it is important to note that this isomorphism is not canonical; it depends on the path p.

Lecture 29 — 11/10/10

Recall that we cannot properly define a map

$$\log : \mathbb{C}^* \to \mathbb{C}$$

since we get, for instance $0 = \log 1 = \log e^{2\pi i} = 2\pi i$. However, $\exp : \mathbb{C} \to \mathbb{C}^*$ is well-defined. We want to formalize the notion that exp *almost* has an inverse, as described before. Moreover, we want to describe the fundamental group $\pi_1(\mathbb{C}^*, 1)$ by considering an action on the set

$$\exp^{-1}\{1\} = \{(2\pi i)n : n \in \mathbb{Z}\}$$

Definition 29.1. Let $f : \widetilde{X} \to X$ be a map of topological spaces. We say that f is a covering map if every point $x \in X$ has a neighborhood U such that

$$f^{-1}(U) \cong \coprod_{s \in S} U_s$$

the disjoint union of open sets U_s, each mapped homeomorphically to U by f. We call $\widetilde{X}$ a covering space.

Note. It follows immediately from this definition that a covering map is a local homeomorphism. Note also that

$$f^{-1}(U) \cong U \times S$$

where S is taken with the discrete topology.

Example. Consider $\exp : \mathbb{C} \to \mathbb{C}^*$. Let

$$U = \{z : |z - 1| < 1\}$$

We can define $\log : U \to \mathbb{C}$ by the formula

$$\log(1 + z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots$$

Write $\log(U) = V \subseteq \mathbb{C}$. Then we get

$$\exp^{-1}(U) = \bigcup_{n \in \mathbb{Z}} (V + 2\pi i n) \cong V \times \mathbb{Z} \cong U \times \mathbb{Z}$$

Observation 29.2. If S has a single element for every x, then f is a homeomorphism. This is true because covering maps are local projections, hence open, and therefore homeomorphic if bijective.

Definition 29.3. Let $f : \widetilde{X} \to X$ be a covering map, let $x \in X$. The fiber of x is the set $f^{-1}\{x\} \subset \widetilde{X}$. Note that all fibers are discrete.

Definition 29.4. Let $\varphi : \widetilde{X} \to X$ be a continuous map of topological spaces. Let $f : Y \to X$ also be a continuous map. A lifting of f is a map $\tilde{f} : Y \to \widetilde{X}$ such that $\varphi \circ \tilde{f} = f$.

$$\begin{tikzcd} & \widetilde{X} \arrow[d, "\varphi"] \\ Y \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

Theorem 29.5. *Let $f : \widetilde{X} \to X$ be a covering map, let $x \in X$, and let $p \in P_{x,x}(X)$. Choose $\tilde{x} \in f^{-1}\{x\}$. Then*

1. *p lifts uniquely to a path $\tilde{p} : [0, 1] \to \widetilde{X}$ such that $\tilde{p}(0) = \tilde{x}$.*
2. *The fundamental group $\pi_1(X, x)$ acts on the fiber $f^{-1}\{x\}$ by the monodromy action,*

$$\tilde{x}[p] = \tilde{p}(1)$$

Proof. For now, we assume (1). To show (2), we want to show the following:

1. $\tilde{p}(1)$ depends only on $[p]$ and $\tilde{x}$, not on p.
2. $\tilde{p}(1) = \tilde{x}$ if p is the constant path $t \mapsto x$.
3. If $[p], [q] \in \pi_1(X, x)$, then $(\tilde{x}[p])[q] = \tilde{x}[p * q]$.

We first prove *2*. Let $1 : [0, 1] \to X$ be the constant path, given by $1(t) = x$. By (1), this lifts uniquely to $\tilde{1} : [0, 1] \to \widetilde{X}$, which is given by $\tilde{1}(t) = \tilde{x}$. Hence,

$$\tilde{x}[1] = \tilde{1}(1) = \tilde{x}$$

Now we prove *3*. Say $p, q \in P_{x,x}(X)$. What is $\widetilde{p * q}$? Let $\tilde{x}' = \tilde{p}(1)$, making $\tilde{p}$ a path from $\tilde{x}$ to $\tilde{x}'$. Let $\tilde{q}'$ be a path from $\tilde{x}'$ to $\tilde{x}''$ (with $f \circ \tilde{q}' = q$). The product $\tilde{p} * \tilde{q}'$ is well-defined, starts on $\tilde{x}$, and does indeed lift $p * q$:

$$\widetilde{p * q} = \tilde{p} * \tilde{q}'$$

Thus, we have,

$$[p * q]\tilde{x} = \tilde{x}'' = \tilde{x}'[q] = (\tilde{x}[p])[q]$$

We leave the proof of *1* and (1) for later. ■

Observation 29.6. Let $f : X \to Y$ be a continuous map between topological spaces, $x \in X$. Then f induces a continuous map $P_{x,x}(X) \to P_{f(x),f(x)}(Y)$ by composition, and hence a map

$$\pi_1(f) : \pi_1(X, x) \to \pi_1(Y, f(x))$$

This is a group homomorphism, called the induced homomorphism, and the proof is left as an exercise.

Observation 29.7. Let $f : \widetilde{X} \to X$ be a covering map, $x \in X$. Choose $\tilde{x} \in f^{-1}\{x\}$, and let $H_{\tilde{x}}$ be its stabilizer under the monodromy action in $\pi_1(X, x)$.

$$H_{\tilde{x}} = \{[p] \in \pi_1(X, x) : \tilde{x}[p] = \tilde{x}\}$$

Equivalently, $\tilde{p}(1) = \tilde{x} = \tilde{p}(0)$. So $H_{\tilde{x}} = \mathrm{im}(\pi_1(f))$, the image of the induced homomorphism

$$\pi_1(f) : \pi_1(\widetilde{X}, \tilde{x}) \to \pi_1(X, x)$$

Observation 29.8. If $\widetilde{X}$ is path-connected, then the monodromy action is transitive. For given any $\tilde{x}, \tilde{x}' \in f^{-1}\{x\}$, there is a path $\tilde{p} : [0, 1] \to \tilde{X}$ from $\tilde{x}$ to $\tilde{x}'$. Hence, $\tilde{x}' = \tilde{x}[f \circ \tilde{p}]$.

Lecture 30 — 11/12/10

Theorem 30.1. *Let X be a topological space, $x \in X$, $f : \widetilde{X} \to X$ a covering map, $p \in P_{x,x}(X)$. Then the fundamental group $\pi_1(X, x)$ acts on $f^{-1}\{x\}$ via the formula*

$$\tilde{x}[p] = \tilde{p}(1)$$

where $\tilde{p} : [0, 1] \to \widetilde{X}$, with $\tilde{p}(0) = \tilde{x}$, lifts p uniquely.

Theorem 30.2 (Homotopy Lifting Property)**.** *Suppose that $f : \widetilde{X} \to X$ is a covering map, K a compact topological space. Let $p : [0, 1] \times K \to X$ be a continuous map. Let $\tilde{p}_0 : K \to \widetilde{X}$ be any map such that*

$$f \circ \tilde{p}_0 = p|_{\{0\}\times K}$$

or, equivalently, such that $\tilde{p}_0$ lifts the map $p_0 : K \to X$ given by $p_0 = p|_{\{0\}\times K}$. Then there exists a unique map

$$\tilde{p} : [0, 1] \times K \to \widetilde{X}$$

such that $\tilde{p}|_{\{0\}\times K} = \tilde{p}_0$, and which lifts p.

Proof. (of Theorem 30.1). Assume that the homotopy lifting property holds. By taking K to be a point, we find that $\tilde{p}$ exists and is unique for every lift $\tilde{x}$ of the point x.

Moreover, we learn that if $p, q \in P_{x,x}(X)$ are homotopic, then $\tilde{p}$ and $\tilde{q}$ are homotopic as well. For suppose that $h : [0, 1] \times [0, 1] \to X$ is a homotopy between paths p and q. Then h lifts uniquely to a map $\tilde{h} : [0, 1] \times [0, 1] \to \widetilde{X}$. So we have

$$f(\tilde{h}(0, t)) = h(0, t) = p(t)$$
$$f(\tilde{h}(1, t)) = h(1, t) = q(t)$$

We know that $\tilde{p}(t) = \tilde{h}(0, t)$, and by uniqueness, we now also have $\tilde{q}(t) = \tilde{h}(1, t)$. Hence, $\tilde{h}$ is indeed a homotopy between $\tilde{p}$ and $\tilde{q}$, so

$$\tilde{x}[p] = \tilde{x}[q]$$

which completes our proof. ∎

Lemma 30.3. *For every $t \in [0, 1]$, there exists an open interval $U \ni t$ such that $\forall t' \in U$ and $\forall \tilde{p}_{t'} : K \to \widetilde{X}$ any lift of the map $p_{t'} : K \to X$ given by $p_{t'} = p|_{\{t'\}\times K}$, $\exists! \tilde{q} : U \times K \to \widetilde{X}$ such that $\tilde{q}$ lifts $p|_{U\times K}$*

$$f \circ \tilde{q} = p|_{U\times K}$$

and $\tilde{q}|_{\{t'\}\times K} = \tilde{p}_{t'}$.

Proof. (of Lemma). Fix $t \in [0, 1]$. For each $a \in K$, $p(t, a) \in X$. Since $f : \widetilde{X} \to X$ is a covering map, we can choose a neighborhood $V_a \ni p(t, a)$ such that

$$f^{-1}(V_a) \cong \coprod V_a$$

By continuity, $p^{-1}(V_a) \supseteq U_a \times W_a$ for some $U_a \overset{\circ}{\subseteq} [0, 1]$, $U_a \ni t$, and some $W_a \overset{\circ}{\subseteq} K$, $W_a \ni a$.

We can write

$$K = \bigcup_{a\in K} W_a$$

By compactness, there are points $a_1, \ldots, a_n \in K$ such that

$$K = W_{a_1} \cup \cdots \cup W_{a_n}$$

Take

$$U := \bigcap_{i=1}^{n} U_{a_i}$$

We know that $U \overset{\circ}{\subseteq} [0, 1]$ and $t \in U$. We claim that U is our desired open set.

Say $t' \in U$, and suppose $\tilde{p}_{t'} : K \to \widetilde{X}$ is given. We want to construct $\tilde{q}$. This problem is local on K; that is, it suffices to construct $\tilde{q}_i = \tilde{q}|_{U\times W_{a_i}}$ for each i. We define $\tilde{q}_i : U \times W_{a_i} \to \widetilde{X}$ by

$$\tilde{q}_i(t', a) = \tilde{p}_{t'}(a)$$

Note that, by construction, we have $f^{-1}(V_{a_i}) \cong \coprod V_{a_i}$ and $p(U \times W_{a_i}) \subseteq V_{a_i}$. Thus, $\tilde{p}_{t'}|_{W_{a_i}}$ must define a map

$$\tilde{p}_{t'}|_{W_{a_i}} : W_{a_i} \longrightarrow f^{-1}(V_{a_i}) \cong V_{a_i} \times S$$

This map is given by $p|_{\{t'\}\times W_{a_i}}$ and some continuous map $\varphi : W_{a_i} \to S$. Hence, our map

$$\tilde{q}_i : U \times W_{a_i} \to V_{a_i} \times S \cong f^{-1}(V_{a_i}) \subseteq \widetilde{X}$$

is determined uniquely by

$$p|_{U\times W_{a_i}} : U \times W_{a_i} \longrightarrow V_{a_i}$$

in one component, and by

$$U \times W_{a_i} \xrightarrow{\pi} W_{a_i} \xrightarrow{\varphi} S$$

in the other.

The maps $\tilde{q}_i$ and $\tilde{q}_j$ agree on $(U \times W_{a_i}) \cap (U \times W_{a_j})$ by uniqueness. Hence, the $\tilde{q}_i$ glue to give a map

$$\tilde{q} : U \times K \to \widetilde{X}$$

with $\tilde{q}(t, a) = \tilde{q}_i(t, a)$ when $a \in W_{a_i}$. The continuity of $\tilde{q}$ is left as an exercise. ∎

Proof. (of Homotopy Lifting Property).
For each $t \in [0, 1]$, choose U_t as in the Lemma. Then

$$[0, 1] = \bigcup_{t\in[0,1]} U_t$$

By compactness, we can choose $t_1, \ldots, t_n \in [0, 1]$ such that

$$[0, 1] = U_{t_1} \cup \cdots \cup U_{t_n}$$

Then we can choose points

$$0 = s_0 \leq s_1 \leq \cdots \leq s_n = 1$$

such that $\forall i \leq n, [s_i, s_{i+1}] \subseteq U_{t_j}$ for some j.

Let $\tilde{p}_0 : K \to \widetilde{X}$ be given. By the Lemma, we can extend $\tilde{p}_0$ to a map

$$\tilde{p}_{0,1} : [s_0, s_1] \times K \to \widetilde{X}$$

such that $\tilde{p}_{0,1}(s_0, a) = \tilde{p}_0(a)$ and $f\tilde{p}_{0,1}(s, a) = p(s, a)$. Then we can choose another map $\tilde{p}_{1,2} : [s_1, s_2] \times K \to \widetilde{X}$ with $\tilde{p}_{1,2}(s_1, a) = \tilde{p}_{0,1}(s_1, a)$ and $f\tilde{p}_{1,2}(s, a) = p(s, a)$. Continuing in this way, we get maps

$$\tilde{p}_{i,i+1} : [s_i, s_{i+1}] \times K \to \widetilde{X}$$

such that

$$\tilde{p}_{i,i+1}(s_i, a) = \tilde{p}_{i-1,i}(s_i, a) \quad \text{and} \quad f\tilde{p}_{i,i+1}(s, a) = p(s, a)$$

We define

$$\tilde{p}(s, a) = \tilde{p}_{i,i+1}(s, a), \qquad \text{where } s \in [s_i, s_{i+1}]$$

By construction, $\tilde{p}(0, a) = \tilde{p}_0(a)$, and by the Lemma, $\tilde{p}$ is unique. The proof that $\tilde{p}$ is continuous is left as an exercise. ■

Let $f : \widetilde{X} \to X$ be a covering map, $x \in X$, and choose $\tilde{x} \in \widetilde{X}$ such that $f(\tilde{x}) = x$. Then we have the induced homomorphism

$$\pi_1(f) : \pi_1(\widetilde{X}, \tilde{x}) \longrightarrow \pi_1(X, x)$$

Recall that

$$\operatorname{im}(\pi_1(f)) = \{[p] \in \pi_1(\widetilde{X}, \tilde{x}) : \tilde{x}[p] = \tilde{x}\}$$

is the stabilizer of $\tilde{x}$.

Proposition 30.4. *$\pi_1(f)$ is injective; equivalently, there is an isomorphism*

$$\pi_1(\widetilde{X}, \tilde{x}) \cong H_{\tilde{x}}$$

where $H_{\tilde{x}}$ is the stabilizer of $\tilde{x}$.

Proof. Let $\tilde{p} \in P_{\tilde{x},\tilde{x}}(\widetilde{X})$, $p = f \circ \tilde{p}$. Assume $[p]$ is trivial in $\pi_1(X, x)$. Then there is a homotopy h for the constant path x and our path p. By the homotopy lifting property, we can lift h to a homotopy $\tilde{h} : [0, 1] \times [0, 1] \to \widetilde{X}$. $\tilde{h}$ is a homotopy from $\tilde{p}$ to the constant path $\tilde{x}$, so

$$[\tilde{p}] \in \pi_1(\widetilde{X}, \tilde{x})$$

is trivial. Thus, $\ker(\pi_1(f))$ is trivial, and hence, $\pi_1(f)$ is injective. ■

Lecture 31 — 11/15/10

Recall that if $f : \widetilde{X} \to X$ is a covering map, then

1. $\pi_1(X, x)$ acts on $f^{-1}\{x\}$.
2. The stabilizer of any given element $\tilde{x} \in f^{-1}\{x\}$ is $\pi_1(\widetilde{X}, \tilde{x})$.

Definition 31.1. A topological space X is simply connected if X is path-connected and $\pi_1(X, x)$ is trivial (a condition which does not depend on x by path-connected).

Observation 31.2. If $\widetilde{X}$ is simply connected, then $\pi_1(\widetilde{X}, \tilde{x}) = 0$ for every $\tilde{x} \in f^{-1}\{x\}$. Hence, the monodromy action is simply transitive, yielding a bijection of sets

$$\pi_1(X, x) \longleftrightarrow f^{-1}\{x\}$$

Example. Consider the covering map

$$\exp : \mathbb{C} \to \mathbb{C}^*$$

The space $\mathbb{C}$ is simply connected since any two paths p and q are homotopic by the homotopy

$$h(s, t) = sp(t) + (1 - s)q(t)$$

Since $\exp^{-1}\{1\} \simeq 2\pi i\mathbb{Z} \simeq \mathbb{Z}$, by the above, we get

$$\pi_1(\mathbb{C}^*, 1) \longleftrightarrow \mathbb{Z}$$

Example. Consider the 1-circle

$$S^1 = \{z \in \mathbb{C} : |z| = 1\}$$

and the covering map given by

$$\begin{aligned} \theta : \mathbb{R} &\longrightarrow S^1 \\ t &\longmapsto e^{2\pi i t} \end{aligned}$$

For any $x \in S^1$, we have

$$\pi_1(S^1, x) \longleftrightarrow \theta^{-1}\{x\} \simeq \mathbb{Z}$$

Remark. One way to obtain covering spaces is via group actions. Let G be a group acting on a topological space $\widetilde{X}$. We want for $X := \widetilde{X}/G$ to be a topological space and

$$\widetilde{X} \longrightarrow \widetilde{X}/G = X$$

a covering map.

Example.

1. Let $\widetilde{X} = \mathbb{R}, G = \mathbb{Z}$. G acts on $\widetilde{X}$ by
$$\begin{aligned} G \times \widetilde{X} &\longrightarrow \widetilde{X} \\ (n, t) &\longmapsto t + n \end{aligned}$$
$X := \widetilde{X}/G \simeq S^1$, and $\widetilde{X} \to X$ is a covering map.

2. Let $\widetilde{X} = \mathbb{C}, G = \mathbb{Z}$. G acts on $\widetilde{X}$ by

$$\begin{aligned} G \times \widetilde{X} &\longrightarrow \widetilde{X} \\ (n, z) &\longmapsto z + 2\pi i n \end{aligned}$$

$X := \widetilde{X}/G \simeq \mathbb{C}^*$, and $\widetilde{X} \to X$ is a covering map.

Remark. An action of G on $\widetilde{X}$ is a map

$$a : G \times \widetilde{X} \to \widetilde{X}$$

such that

1. $a(1, \tilde{x}) = \tilde{x}$.
2. $a(g, a(g', \tilde{x})) = a(gg', \tilde{x})$.

We want group action to respect topology in some way; that is, we want it to be continuous. Equivalently, $\forall g \in G$, we want the map $\tilde{x} \mapsto a(g, \tilde{x})$ to be continuous.

Definition 31.3. Let Y, Y' be topological spaces and let $p : Y \twoheadrightarrow Y'$. We call p a <u>quotient map</u> provided that $U \mathring{\subseteq} Y'$ iff $p^{-1}(U) \mathring{\subseteq} Y$. Note that this is a strictly stronger condition than continuity.

Definition 31.4. Let Y be a topological space, A any set. Let $p : Y \twoheadrightarrow A$. There is a single topology on A relative to which p is a quotient map; this topology is called the <u>quotient topology</u>, and is given by $U \mathring{\subseteq} A$ if $p^{-1}(U) \mathring{\subseteq} Y$.

Note. Suppose that $\{U_\alpha\}$ are open in A. Then $p^{-1}(\bigcup_\alpha U_\alpha) = \bigcup_\alpha p^{-1}(U_\alpha)$, so $\bigcup U_\alpha$ is open. The same is true for finite intersections.

Definition 31.5. Let Y be any topological space, $\sim$ be an equivalence relation on Y. Consider the map

$$\begin{aligned} p : Y &\longrightarrow Y/\sim \\ y &\longmapsto \overline{y} \end{aligned}$$

Equipping $Y/\sim$ with the quotient topology w.r.t. p yields a <u>quotient space</u>.

Note. Let Z be any set. Then giving a map $f : Y/\sim \to Z$ is equivalent to giving a map $f' : Y \to Z$ such that

1. $f'(a) = f'(b)$ when $a \sim b$.
2. $f(x) = f'(\overline{x})$.
3. $f' = f \circ p$.

Claim 31.6. *If Z has a topology, f is continuous iff f' is continuous.*

Remark. Returning to our discussion of covering spaces, we find that $\widetilde{X}/G$ inherits the quotient topology, and the map $\widetilde{X} \to \widetilde{X}/G$ is continuous. However, it is not usually a covering map.

Example. Let $\widetilde{X} = \mathbb{R}, G = \mathbb{Q}$. G acts on $\widetilde{X}$ by

$$\begin{aligned} G \times \widetilde{X} &\longrightarrow \widetilde{X} \\ (q, t) &\longmapsto t + q \end{aligned}$$

$\mathbb{R}/\mathbb{Q}$ inherits the quotient topology. We claim that this is the trivial topology.

We know there is a bijection between open subsets of $\mathbb{R}/\mathbb{Q}$ and open subsets of $\mathbb{R}$ that are invariant under translation by $\mathbb{Q}$. But there are no such proper subsets of $\mathbb{R}$. For suppose that $U \mathring{\subseteq} \mathbb{R}$ is invariant. If $U \neq \emptyset$, then $U \supseteq (t - \epsilon, t + \epsilon)$ for some $t \in \mathbb{R}, \epsilon > 0$.

Choose $q \in \mathbb{Q}$ such that $q \in (t - \frac{\epsilon}{2}, t + \frac{\epsilon}{2})$. For any $t' \in \mathbb{R}$, we can choose $q' \in \mathbb{Q}$ such that $|q' - t'| < \frac{\epsilon}{2}$. Then $t' - q' + q \in (t - \epsilon, t + \epsilon) \subseteq U$. Hence, $t' \in U$, so $U = \mathbb{R}$.

This does not yield a covering map.

Definition 31.7. Let G be a group acting on a topological space $\widetilde{X}$. We say the action is <u>topologically free</u> or <u>properly discontinuous</u> if $\forall x \in \widetilde{X}$, there is $U \mathring{\subseteq} \widetilde{X}$ such that $U \ni x$ but

$$\forall g \in G, g \neq 1, U \cap gU = \emptyset$$

where $gU = \{y \in \widetilde{X} : y = gy' \text{ for } y' \in U\}$.

Example. $\mathbb{Z} \circlearrowleft \mathbb{R}$ is topologically free, since

$$\left((-\frac{1}{2}, \frac{1}{2}) + n\right) \cap \left(-\frac{1}{2}, \frac{1}{2}\right) = \emptyset$$

But $\mathbb{Q} \circlearrowleft \mathbb{R}$ is not topologically free.

Theorem 31.8. *Let G be a group with a topologically free action on a space $\widetilde{X}$. Let $X = \widetilde{X}/G$. Then the map $p : \widetilde{X} \to X$ is a covering map.*

Proof. Let $x \in X$. Choose $\tilde{x} \in \widetilde{X}$ such that $p(\tilde{x}) = x$. Choose $U \mathring{\subseteq} \widetilde{X}, U \ni \tilde{x}$ such that $\forall g \neq 1, U \cap gU = \emptyset$. We claim that $p(U)$ is an open subset of x.

We prove a more general claim, that $p : \widetilde{X} \to \widetilde{X}/G$ is an open map. If $V \mathring{\subseteq} \widetilde{X}$, then

$$p^{-1}(p(V)) = \bigcup_{g \in G} gV$$

which is a union of open sets, and hence is open. So $p(V) \mathring{\subseteq} \widetilde{X}/G$ since p is a quotient map.

We claim that $p^{-1}(p(U)) \subseteq \widetilde{X}$ is an even covering of $p(U) \ni x$. Note that

$$p^{-1}(p(U)) = \bigcup_{g \in G} gU = \coprod_{g \in G} gU$$

We claim that the map $gU \to p(U)$ is a homeomorphism. We know that gU contains at most one point of each orbit, else gU would not be disjoint from each $g'U$. Hence, gU and $p(U)$ are in bijection. But we also know that p is continuous and open. Thus, $gU \cong p(U)$, and hence p is indeed a covering map, as desired. ∎

Observation 31.9.

1. The action of G is simply transitive. For consider gU and $g'U$; the element $g'g^{-1} \in G$ takes gU to $g'U$, and it is unique by proper discontinuity. It follows that g acts as a homeomorphism on $\widetilde{X}$.

2. The action of G and the monodromy action commute
$$(g\tilde{x})[q] = \tilde{q}_{g\tilde{x}}(1) = (g \circ \tilde{q}_{\tilde{x}})(1) = g(\tilde{x}[q])$$
since $\tilde{q}_{g\tilde{x}}$ and $g \circ \tilde{q}_{\tilde{x}}$ are both lifts of q starting at $\tilde{x}$, and hence are unique.

3. By our first observation, given $\tilde{x} \in \widetilde{X}$, $x = p(\tilde{x})$, we get
$$p^{-1}\{x\} = \{g\tilde{x} : g \in G\} \simeq G$$
Hence, the monodromy action of $\pi_1(X, x)$ gives a map
$$\varphi : \pi_1(X, x) \longrightarrow G$$
which takes a path $[q]$ to the element $g_q \in G$ such that $g_q\tilde{x} = \tilde{x}[q]$. Then this map φ is a homomorphism, for we have
$$\begin{aligned} \varphi([q * q'])\tilde{x} &= \tilde{x}[q * q'] \\ &= \tilde{x}[q][q'] \\ &= \varphi([q'])\tilde{x}[q] \\ &= \varphi[q]\varphi([q'])\tilde{x} \end{aligned}$$
If $\widetilde{X}$ is path-connected, then the monodromy action is transitive, so this map is surjective. If $\widetilde{X}$ is simply connected, then the monodromy action is simply transitive, so this map is an isomorphism,
$$\pi_1(X, x) \simeq G \simeq p^{-1}\{x\}$$

Example. $\pi_1(\mathbb{C}^*, 1) \simeq \mathbb{Z}$ is a group.

Lecture 32 — 11/17/10

Definition 32.1. Let X be any topological space. A universal cover of X is a simply connected space that covers X.

Definition 32.2. Let X be a topological space. X is semilocally path-connected if $\forall x \in X$, $\forall U \overset{\circ}{\subseteq} X : U \ni x$, $\exists V \overset{\circ}{\subseteq} X : V \subseteq U, V \ni x$ such that every point $y \in V$ is connected to x by a path in U.

Claim 32.3. *Let X be a topological space. TFAE:*

1. *X is semilocally path-connected.*

2. *$\forall U \overset{\circ}{\subseteq} X$, the path components of U are open.*

Proof. The proof is trivial. ∎

Observation 32.4. If X is semilocally path-connected, so is any covering space of X, since both of these are local properties. Also note that path-connectedness and connectedness are equivalent in a semilocally path-connected space, as are their respective components.

Proposition 32.5. *Let X be a topological space that is both semilocally path-connected and simply connected. Then every covering space of X is homeomorphic to $X \times S$ for some set S.*

Proof. Let $f : \widetilde{X} \to X$ be a covering map. Note that $\widetilde{X}$ is a disjoint union of open path components by semilocal path-connectedness. So WLOG, assume that $\widetilde{X}$ is path-connected. Then the monodromy action of $\pi_1(X, x)$ is transitive. But $\pi_1(X, x) = 0$, so $f^{-1}\{x\}$ has only one point. Hence, $\widetilde{X} \cong X$. ∎

Corollary 32.6. *Let X be semilocally path-connected. Let $\widetilde{X}$ be a universal cover of X with covering map*
$$f : (\widetilde{X}, \tilde{x}) \to (X, x)$$
Then for any covering map $f' : \widetilde{X}' \to X$ and any point $\tilde{x}' \in f'^{-1}\{x\}$, there is a unique map $g : \tilde{X} \to \widetilde{X}'$ such that $g(\tilde{x}) = \tilde{x}'$ and the following diagram is commutative

$$\begin{tikzcd} \widetilde{X} \arrow[d, "g"'] \arrow[dr, "f"] & \\ \widetilde{X}' \arrow[r, "f'"'] & X \end{tikzcd}$$

Proof. Consider the fiber product
$$Y = \widetilde{X} \times_X \widetilde{X}' = \{(a, b) : a \in \widetilde{X}, b \in \widetilde{X}', f(a) = f'(b)\}$$
The projection map $\tilde{f} : Y \to \widetilde{X}$ is a covering map. Since $\widetilde{X}$ is simply connected, $Y \cong \widetilde{X} \times S$, and in particular we find that $S \cong f^{-1}\{x\}$.

Note that giving a map $g : \widetilde{X} \to \widetilde{X}'$ such that $f' \circ g = f$ is equivalent to giving a map $g' : \widetilde{X} \to Y$ such that $\tilde{f} \circ g' = \mathrm{id}$. This is the same as giving a continuous map $\widetilde{X} \to S$, which, since $\widetilde{X}$ is simply connected, is constant and is the same as giving an element of S. ∎

Corollary 32.7. *Let X be semilocally path-connected, $f : (\widetilde{X}, \tilde{x}) \to (X, x)$ a universal covering map. For every element $\tilde{x}' \in f^{-1}\{x\}$, there is a unique map $f' : \widetilde{X} \to \widetilde{X}$ such that $f \circ f' = f$ and $f'(\tilde{x}) = \tilde{x}'$.*

Note. By applying the corollary with $\tilde{x}$ and $\tilde{x}'$ interchanged, we find an inverse to f', and hence all such maps f' are homeomorphisms from $\widetilde{X} \to \widetilde{X}$.

Definition 32.8. Let $f : \widetilde{X} \to X$ be a covering map, and define

$$\operatorname{Aut}(f) = \{\gamma : \widetilde{X} \to \widetilde{X} \mid \gamma \text{ homeomorphic, } f \circ \gamma = f\}$$

Each such γ is called a deck transformation or covering transformation. Moreover, since inverses and compositions of homeomorphisms are themselves homeomorphims, $\operatorname{Aut}(f)$ is a group, called the group of deck transformations of the given cover. This group $\operatorname{Aut}(f)$ has a natural action on $\widetilde{X}$ by evaluation.

Lemma 32.9 (Unique Lifting Property)**.** *Let X, Y be arbitrary topological spaces, $f : (\widetilde{X}, \tilde{x}) \to (X, x)$ a covering map, $g : (Y, y_0) \to (X, x)$ a continuous map with a lifting $\tilde{g} : (Y, y_0) \to (\widetilde{X}, \tilde{x})$; that is, we have*

$$\begin{tikzcd} & \widetilde{X} \arrow[d, "f"] \\ Y \arrow[ur, "\tilde{g}"] \arrow[r, "g"'] \arrow[ur, phantom, "\circlearrowleft"', shift right=2ex] & X \end{tikzcd}$$

If Y is connected, then this lifting is unique.

Proof. Let $\tilde{g}' : (Y, y_0) \to (\widetilde{X}, \tilde{x})$ be another lifting. Let $y \in Y$. Choose an open neighborhood $U \ni x$ such that $f^{-1}(U) = \coprod \widetilde{U}_\alpha$ evenly covers U. Take $\widetilde{U} \ni \tilde{g}(y)$ and $\widetilde{U}' \ni \tilde{g}'(y)$ among these disjoint open sets. By continuity, $\exists V \overset{\circ}{\subseteq} Y$, $V \ni y$ such that $\tilde{g}(V) \subseteq \widetilde{U}$ and $\tilde{g}'(V) \subseteq \widetilde{U}'$.

If $\tilde{g}(y) \neq \tilde{g}'(y)$, then $\widetilde{U} \neq \widetilde{U}'$. Then $\widetilde{U} \cap \widetilde{U}' = \emptyset$, and hence $\tilde{g}(y') \neq \tilde{g}'(y')$ for any $y' \in V$. If instead $\tilde{g}(y) = \tilde{g}'(y)$, then $\widetilde{U} = \widetilde{U}'$. Then we must have $\tilde{g}|_V = \tilde{g}'|_V$ since $f \circ \tilde{g}|_V = f \circ \tilde{g}'|_V$ and f is injective from $\widetilde{U} = \widetilde{U}'$ into U. So the set $\{y \in Y : \tilde{g}(y) = \tilde{g}(y')\}$ is clopen, and hence is all of Y. ∎

Observation 32.10. Let $\widetilde{X}$ be path-connected. Choose $x \in X$ and $\tilde{x} \in f^{-1}\{x\}$. Each deck transformation $\gamma \in \operatorname{Aut}(f)$ is a lift of the covering map f, and hence is uniquely determined by its action on $\tilde{x}$. Thus, the action of $\operatorname{Aut}(f)$ is free (though not necessarily transitive).

Observation 32.11. Recall Theorem 31.8, where we showed that if G is a topologically free action on $\widetilde{X}$ and $X = \widetilde{X}/G$, then $f : \widetilde{X}/G \to X$ is a covering map. We now claim that if $\widetilde{X}$ is path-connected, then $G = \operatorname{Aut}(f)$.

Choose $\gamma \in \operatorname{Aut}(f)$. Then for any point $\tilde{x} \in \widetilde{X}$, there exists $g \in G$ taking $\tilde{x}$ to $\gamma(\tilde{x})$ (this is true because X is the orbit space and f the canonical projection). But then $g(\tilde{x}) = \gamma(\tilde{x})$, which yields $g = \gamma$ by freeness, and hence $G = \operatorname{Aut}(f)$.

Proposition 32.12. *Let $f : \widetilde{X} \to X$ be a covering map, $\widetilde{X}$ path-connected, G the group of deck transformations. Then the action of G is topologically free.*

Proof. Let $U \overset{\circ}{\subseteq} X$ be evenly covered, with $\widetilde{U} \cong U$ in this even covering. Let $g \in G$. If $\widetilde{U} \cap g\widetilde{U} \neq \emptyset$, then $\exists \tilde{x}, \tilde{x}' \in \widetilde{X}$ such that $\tilde{x} = g\tilde{x}'$. Since g is a deck transformation, $\tilde{x}$ and $\tilde{x}'$ are in the same fiber $f^{-1}\{x\}$. But this fiber intersects $\widetilde{U}$ at only one point, so $\tilde{x} = \tilde{x}'$. Hence, by uniqueness, $g = 1$, so our action is indeed topologically free. ∎

Proposition 32.13. *If, in addition, X is path-connected and the action of G is transitive (and hence simply transitive), then $\tilde{X}/G \cong X$.*

Proof. We have

$$\begin{tikzcd} \widetilde{X} \arrow[d, "\pi"'] \arrow[dr, "f"] \arrow[dr, phantom, "\circlearrowleft"', shift right=2ex] & \\ \widetilde{X}/G \arrow[r, "\varphi"'] & X \end{tikzcd}$$

We want to show that φ is a homeomorphism. Since f is surjective, so is φ. Moreover, since G acts simply transitively, φ is bijective. It suffices to show that φ is open to complete the proof. If $U \subseteq \widetilde{X}/G$ is open, then $\pi^{-1}(U)$ is open, and $\varphi(U) = (\varphi \circ \pi)(\pi^{-1}(U))$. But $\varphi \circ \pi = f$ is a covering map, and hence open, as desired. ∎

Lecture 33 — 11/19/10

Recall that a topological space X is called semilocally path-connected if $\forall x \in U \overset{\circ}{\subseteq} X$, $\exists V \overset{\circ}{\subseteq} X$, $V \subset U$, $V \ni x$ such that every point $y \in V$ is connected to x by a path in U.

Definition 33.1. A topological space X is semilocally simply connected if $\forall x \in X$, $\exists U \overset{\circ}{\subseteq} X, U \ni x$ such that the map $\pi_1(U, y) \to \pi_1(X, y)$ induced by inclusion is trivial for all $y \in U$.

Theorem 33.2. *If X is path-connected, semilocally path-connected, and semilocally simply connected, then X has a universal cover.*

Proof. Fix $x \in X$. Let

$$P_x := \{p \in \operatorname{Map}([0,1], X) : p(0) = x\}$$

Let $\sim$ be an equivalence relation on P_x such that $p \sim q$ if $p(1) = q(1)$ and $[p] = [q]$. Define

$$\widetilde{X} := P_x / \sim$$

We give $\operatorname{Map}([0,1], X)$ the compact-open topology, P_x the subspace topology, and $\widetilde{X}$ the quotient topology. Let $\pi : P_x \to \widetilde{X}$ be the quotient map, $\bar{f} : P_x \to X$ the map given by $\bar{f}(p) = p(1)$, and $f : \widetilde{X} \to X$ the map induced via π; that is,

$$\begin{tikzcd} P_x \arrow[d, "\pi"'] \arrow[dr, "\bar{f}"] & \\ \widetilde{X} \arrow[r, "f"'] \arrow[ur, phantom, "\circlearrowleft"] & X \end{tikzcd}$$

We will show that f is a covering map and that $\widetilde{X}$ is simply connected.

First we will show that f is open. Let $U \mathring{\subseteq} \widetilde{X}$. $\pi^{-1}(U) \subseteq P_x$ is open by continuity. Then

$$f(U) = \bar{f}(\pi^{-1}(U))$$

So it suffices to show that $\bar{f}$ is open.

Say $V \mathring{\subseteq} P_x$. Say $y \in \bar{f}(V)$, so $y = p(1)$ for some $p \in V$ (this is true by surjectivity, which follows from path-connectedness). We want to find a neighborhood of y contained in $\bar{f}(V)$. Since V is open, there exist compact sets $K_1, \dots, K_n \subseteq [0,1]$ and open sets $U_1, \dots, U_n \mathring{\subseteq} X$ such that we have

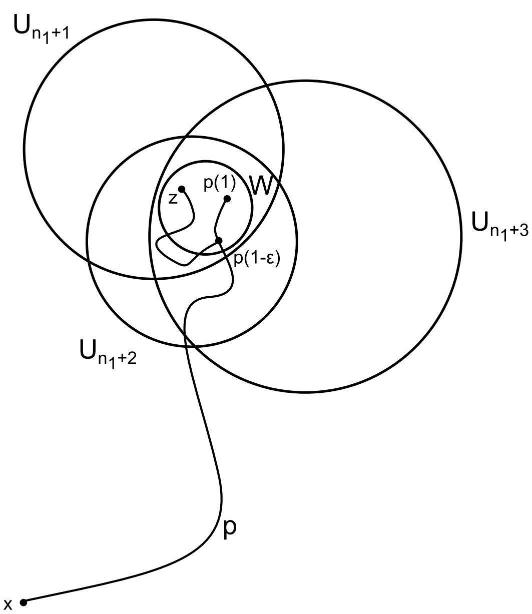
$$p \in \{q \in P_x : \forall i \leq n, q(K_i) \subseteq U_i\} \subseteq V$$

WLOG, write $n = n_1 + n_2$, where $K_1, \dots, K_{n_1} \not\ni 1$ and where $K_{n_1+1}, \dots K_n \ni 1$. So there is $\epsilon > 0$ such that $K_1, K_2, \dots, K_{n_1} \subseteq [0, 1-\epsilon]$. Hence,

$$p(1) \in U_{n_1+1} \cap \cdots \cap U_n$$

By semilocal path-connectedness, $\exists W \mathring{\subseteq} X : W \ni p(1)$, $W \subseteq U_{n_1+1} \cap \cdots \cap U_n$ such that points in W are connected by paths in $U_{n_1+1} \cap \cdots \cap U_n$.

We claim that $\bar{f}(V) \supseteq W$. WLOG, choose ϵ small enough that $p([1-\epsilon]) \subseteq W$. Then for any $z \in W$, we can construct a new path from x to z which stays in our basic open subset of V as follows:



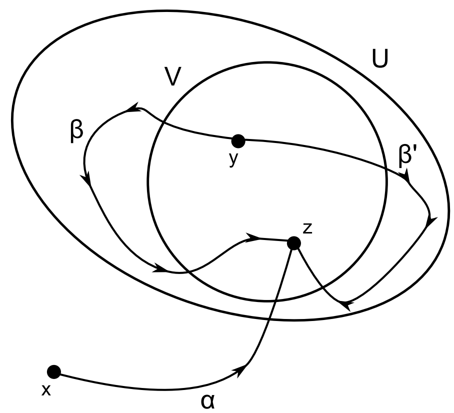
This means that $p(1) = y \in W \subseteq \bar{f}(V)$, so $\bar{f}$ is open, as is f.

Next, we claim that f is a covering map. Choose $y \in X$. We want an open neighborhood $V \ni y$ such that $f^{-1}(V) \cong \coprod V$. Choose $U \ni y$ such that $\pi_1(U) \to \pi_1(X)$ is trivial. Then choose $V \ni y$, $V \subseteq U$ such that all points in V are connected by paths in U. We claim that V is our desired neighborhood.

For each $[\gamma] \in \pi_0 P_{x,y}$—that is to say, for every homotopy class of paths from x to y—let $V_{[\gamma]} \subseteq \widetilde{X}$ be the collection homotopy classes $[\gamma * \beta]$, where β is some path in U such that $\beta(0) = \gamma(1)$ (necessarily) and $\beta(1) \in V$. To complete the proof, we must show that

1. $f^{-1}(V) = \bigcup_{[\gamma] \in \pi_0 P_{x,y}} V_{[\gamma]}$.
2. $V_{[\gamma]} \cap V_{[\gamma']} = \emptyset$ if $[\gamma] \neq [\gamma']$.
3. $V_{[\gamma]}$ is open.
4. $V_{[\gamma]} \cong V$ via f.

We begin by proving *1* and *2*, and in doing so, will consider the following figure:



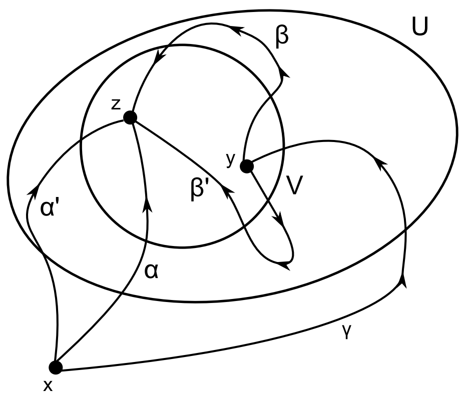
where α is a path starting at x and ending at some point $z \in V$, and hence a representative of some homotopy class $[\alpha] \in f^{-1}(V)$. The inclusion $f^{-1}(V) \supseteq V_{[\gamma]}$ is obvious; we must show $f^{-1}(V) \subseteq \bigcup V_{[\gamma]}$. Take $\gamma = \alpha * \bar{\beta}$, where $\bar{\beta}$ denotes the reverse of β, we get $[\alpha] \in V_{[\gamma]}$. This proves *1*.

To prove *2*, we want to show that our choice of $[\gamma]$ for each $[\alpha]$ is unique. It is obvious that any two representatives of $[\alpha]$ concatenated with some $\bar{\beta}$ yields a singular $[\gamma]$. Now consider a single α representing $[\alpha]$ and β, β' as pictured above. Write $\gamma = \alpha * \bar{\beta}$ and $\gamma' = \alpha * \bar{\beta}'$. Then we have

$$\begin{aligned} \bar{\gamma}' * \gamma &= \beta' * \bar{\alpha} * \alpha * \bar{\beta} \\ &\sim \beta' * \bar{\beta} \end{aligned}$$

This is a loop in U, and by semilocal simply connectedness, it is nullhomotopic. So $\gamma \sim \gamma'$, as desired.

Now assume *3*, and we will prove *4*. Fix $[\gamma]$, and consider the restriction $f_{[\gamma]} : V_{[\gamma]} \to V$. This map inherits continuity and openness from f; hence, it is a homeomorphism iff it is bijective. By path-connectedness of V, $f_{[\gamma]}$ is surjective. To show injectivity, consider the paths given by



We have $\alpha \sim \gamma * \beta$ and $\alpha' \sim \gamma * \beta'$, and by the same argument as before, by semilocal simply connectedness, $\alpha \sim \alpha'$, and hence $[\alpha], [\alpha'] \mapsto z \in V \iff [\alpha] = [\alpha']$.

Finally, we want to show *3*, that $V_{[\gamma]}$ is open in $\widetilde{X}$. This is equivalent to showing that $\pi^{-1}(V_{[\gamma]})$ is open in P_x, since π is the natural quotient map. Say $p \in \pi^{-1}(V_{[\gamma]})$. Then $p(1) \in V$, $p \sim \gamma * \beta$ for a path β contained in U. We want to show that any sufficiently close path satisfies the same conditions.

For each $t < 1$, choose an open set $U_t \ni p(t)$ such that $\pi_1(U_t) \to \pi_1(X)$ is trivial, and we have $U_1 = U$. The sets $\{p^{-1}(U_t)\}$ cover $[0, 1]$, and by compactness, admit a finite subcover $\{p^{-1}(U_{t_j})\}$. We can choose N sufficiently large such that $[\frac{i}{N}, \frac{i+1}{N}] \subseteq p^{-1}(U_{t_j})$ for some j, depending on i. For a given i, write

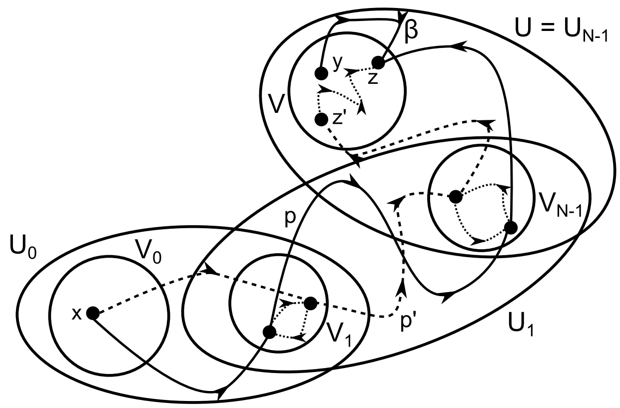
$$U_i := p(p^{-1}(U_{t_j})) = U_{t_j}$$

noting that each U_i is an open set satisfying the semilocal simply connectedness condition and that WLOG, $U_{N-1} = U$. Note also that the U_i may not necessarily be distinct, but this fact is immaterial to our proof.

For $i > 0$, we have $p(\frac{i}{N}) \in U_{i-1} \cap U_i$, and hence there is a subset V_i of this intersection whose points are connected by paths in $U_{i-1} \cap U_i$ (for $i = 0$, we make a similar choice using the open set U_0). Finally, define

$$W = \left\{ q \in P_x : q\left([\frac{i}{N}, \frac{i+1}{N}] \right) \subseteq U_i, q(\frac{i}{N}) \in V_i \right\}$$

W is a basic open set and $W \ni p$. It remains to be shown that $W \subseteq \pi^{-1}(V_{[\gamma]})$. The details of this proof is left as an exercise; a picture proof is given:



Finally, it remains to be shown that $\widetilde{X}$ is simply connected. First we show that it is path-connected. Choose $[\gamma] \in \widetilde{X}$ and let γ_t be the path obtained from the restriction $\gamma|_{[0,t]}$. The map $t \mapsto [\gamma_t]$ is a path in $\widetilde{X}$ from $[x]$ (the nullhomotopy class at x) to $[\gamma]$ that lifts γ. Hence, $\widetilde{X}$ is path-connected.

Now we show that $\pi_1(\widetilde{X}, [x]) = 0$. Recall the monodromy action $f^{-1}\{x\} \circlearrowleft \pi_1(X, x)$, which is transitive by path-connectedness. But by our definition of $\widetilde{X}$, $f^{-1}\{x\} = \pi_1(X, x)$. Then the stabilizers of the monodromy action must all be trivial; that is, this action must be simply transitive. But this means that $\widetilde{X}$ is simply connected, as desired. ∎

Lecture 34 — 11/22/10

Recall that if X is a topological space that is path-connected, semilocally path-connected, and semilocally simply connected, then X has a universal cover $f : \widetilde{X} \to X$. $\widetilde{X}$ has an action of $\pi_1(X, x)$ by concatenation, which is topologically free, yielding $\widetilde{X}/\pi_1(X, x) = X$.

Claim 34.1. *If X has a universal cover $f : \widetilde{X} \to X$, then X is semilocally simply connected.*

***Proof*.** For each $x \in X$, choose a neighborhood $V \ni x$ such that $f^{-1}(V) = \coprod V_\alpha$. Since f maps V_α to V homeomorphically, a loop about x lifts to a loop in $V_\alpha \ni \tilde{x}$ about some specified $\tilde{x}$. So the monodromy action of $\pi_1(V, x)$ on $f^{-1}\{x\} \subseteq f^{-1}(V)$ is trivial. But the monodromy action of $\pi_1(X, x)$ is simply transitive, and hence the homomorphism $\pi_1(V, x) \to \pi_1(X, x)$ induced by inclusion is indeed trivial.

Taking a more visualizable approach, we know that the lift to $\widetilde{X}$ described above is nullhomotopic by simply connectedness. Composition with f yields a nullhomotopy in X of any path, which also provides for semilocal simply connectedness. ∎

Definition 34.2. Let $n \in \mathbb{N}$. A topological space X is an n-manifold if $\forall x \in X$, there is a neighborhood $U \ni x$ such that $U \cong \mathbb{R}^n$.

Example.

1. $\mathbb{R}^n$ is an n-manifold.
2. The n-sphere S^n is an n-manifold.
3. The torus $S^1 \times S^1$ is a 2-manifold. Any tori of higher genus (greater number of holes) are also 2-manifolds.

Observation 34.3. Recall that if X is a topological space, $x \in X$, then by the monodromy action, we have a map from the collection of covering spaces of X into the collection of sets acted on by $\pi_1(X, x)$:

$$(f : \widetilde{X} \to X) \longmapsto f^{-1}\{x\}$$

We claim that if X is path-connected, semilocally path-connected, and semilocally simply connected, we can reverse this construction to yield an equivalence of categories.

Let $f : \widetilde{X} \to X$ be a universal cover of X; since the universal cover is unique up to isomorphism of covering spaces, we take it to be the cover previously constructed. Let S be a set with a right action of $\pi_1(X, x)$. The fundamental group $\pi_1(X, x)$ also has an action on $\widetilde{X}$: for $[\gamma] \in \widetilde{X}$ and $[\lambda] \in \pi_1(X, x)$, we take $[\lambda][\gamma] = [\lambda * \gamma]$. Then consider the map

$$g : (\widetilde{X} \times S)/\pi_1(X, x) \longrightarrow X$$

which takes the orbit of $([\gamma], \widetilde{x})$ to $\gamma(1) \in X$. Note that this gives

$$g^{-1}\{x\} = (f^{-1}\{x\} \times S)/\pi_1(X, x) \cong S$$

We claim that g is a covering map. Choosing $U \subseteq X$ such that $f^{-1}(U) \cong U \times \pi_1(X, x)$, we get $g^{-1}(U) \cong U \times S$, as desired.

Observation 34.4. By similar means, we can show that the collection of all path-connected covering spaces of X maps to the subgroups $H \leq \pi_1(X, x)$ and to sets of the form $\pi_1(X, x)/H$ (the coset space) with an action of $\pi_1(X, x)$.

For each such subgroup H, we get a covering space $\widetilde{X}/H \to X$ with

$$H \simeq \pi_1(\widetilde{X}/H, \tilde{x}) \longrightarrow \pi_1(X, x) \circlearrowleft \pi_1(X, x)/H$$

where the action is the monodromy action; this yields a bijection and an equivalence of categories, as above. For more details, see Munkres 54.6 and Hatcher 1.36.

Example. $\mathbb{R} \to S^1 \cong \mathbb{R}/\mathbb{Z}$ is a covering map, and our above observation yields $\pi_1(S^1, *) \simeq \mathbb{Z}$ for any point $* \in S^1$.

Let us now consider the problem of computing the fundamental group of the doubly-punctured complex plane with any basepoint,

$$\pi_1(\mathbb{C} - \{0, 1\}, x)$$

We notice that $\mathbb{C} - \{0, 1\}$ can be neatly represented by the gluing of two simpler spaces,

$$A = \{x + yi : x < 1\} - \{0\} \cong \mathbb{C} - \{0\}$$
$$B = \{x + yi : x > 0\} - \{1\} \cong \mathbb{C} - \{1\}$$

Note that the intersection

$$A \cap B = \{x + yi : 0 < x < 1\} \cong \mathbb{C}$$

is an even simpler space. We want to reduce the computation of $\pi_1(\mathbb{C} - \{0, 1\}, x)$ to the computation of $\pi_1(\mathbb{C}^*, x)$.

Theorem 34.5 (Seifert-van Kampen Theorem)**.** *Let X be a space covered by two open sets $U, V \overset{\circ}{\subseteq} X$ where $U, V, U \cap V$ are all path-connected. Choose a basepoint $x \in U \cap V$. Then the diagram*

$$\begin{tikzcd} \pi_1(U \cap V, x) \arrow[r] \arrow[d] & \pi_1(U, x) \arrow[d] \\ \pi_1(V, x) \arrow[r] & \pi_1(X, x) \end{tikzcd}$$

is a pushout of groups. This determines $\pi_1(X, x)$ as the free product with amalgamation of $\pi_1(U, x)$ and $\pi_1(V, x)$. Alternatively, this means that for every group G, giving a group homomorphism $\pi_1(X, x) \to G$ is equivalent to giving a pair of maps $\pi_1(U, x) \to G$, $\pi_1(V, x) \to G$ such that the composite maps $\pi_1(U \cap V, x) \to G$ are the same.

Example. Let $X = \mathbb{C} - \{0, 1\}$, $U = \{z \in X : \Re(z) < 1\}$, $V = \{z \in X : \Re(z) > 0\}$. We know $\pi_1(U, \frac{1}{2}) \simeq \mathbb{Z}$, $\pi_1(V, \frac{1}{2}) \simeq \mathbb{Z}$, and $\pi_1(U \cap V, \frac{1}{2}) = 0$. This determines $\pi_1(X, \frac{1}{2})$ as the free group with two generators.

Example. Let $X = S^2$. Choose $x, y, z \in S^2$ all distinct, and take $U = S^2 - \{y\} \cong \mathbb{R}^2$, $V = S^2 - \{z\} \cong \mathbb{R}^2$. We have $\pi_1(U, x) = \pi_1(V, x) = 0$, and $\pi_1(U \cap V, x) \simeq \mathbb{Z}$. Then our pushout yields $\pi_1(X, x) = 0$, so S^2 is simply connected.

Definition 34.6. Recall that $\pi_1(X, x) = \pi_0 P_{x,x}$. Define

$$\pi_2(X, x) = \pi_1(P_{x,x}, 1)$$

Example. $\pi_2(S^2, x) \simeq \mathbb{Z}$.

Lecture 35 — 11/24/10

Theorem 35.1 (Seifert van-Kampen Theorem)**.** *Let X be a topological space, $X = U_1 \cup U_2$ for $U_1, U_2 \overset{\circ}{\subseteq} X$ and $U_1, U_2, U_1 \cap U_2$ all path-connected, $x \in U_1 \cap U_2$, G any group. Then for any pair of group homomorphisms*

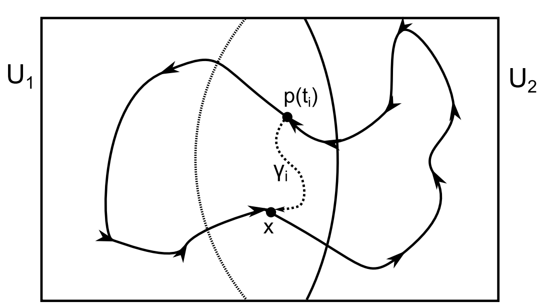
$$\varphi_1 : \pi_1(U_1, x) \to G \qquad \text{and} \qquad \varphi_2 : \pi_1(U_2, x) \to G$$

inducing the same homomorphism $\pi_1(U_1 \cap U_2, x) \to G$, there is a unique homomorphism $\varphi : \pi_1(X, x) \to G$ compatible with φ_1 and φ_2. Equivalently, there is a pushout diagram

$$\begin{tikzcd}
& \pi_1(U_1,x) \arrow[d, "j_1"] \arrow[dr, "\varphi_1"] & \\
\pi_1(U_1 \cap U_2, x) \arrow[ur, "i_1"] \arrow[r] \arrow[dr, "i_2"'] & \pi_1(X,x) \arrow[r, dotted, "\varphi"] & G \\
& \pi_1(U_2,x) \arrow[u, "j_2"'] \arrow[ur, "\varphi_2"'] &
\end{tikzcd}$$

where i_1, i_2, j_1, j_2 are the homomorphisms induced by the relevant inclusions.

Proof. We will describe a construction of our homomorphism $\varphi : \pi_1(X, x) \to G$. Choose a loop $p : [0,1] \to X$ about x. For each $t \in [0,1]$, $p(t) \in U_i$ for some i, so $p(t-\epsilon, t+\epsilon) \subseteq U_i$ for small ϵ. By compactness, we can choose a sequence $0 = t_0 < t_1 < \ldots < t_n = 1$ such that $p(t_i) \in U_1 \cap U_2$ and $p([t_i, t_{i+1}]) \subseteq U_{j(i)}$, $j(i) \in \{1, 2\}$. Choose for each $0 < i < n$ a path γ_i from $p(t_i)$ to x which stays in $U_1 \cap U_2$.



Define

$$\varphi([p]) = \varphi_{j(0)}([p|_{[0,t_1]} * \gamma_1]) * \varphi_{j(1)}([\bar{\gamma}_1 * p|_{[t_1,t_2]} * \gamma_2]) * \cdots * \varphi_{j(n-1)}([\bar{\gamma}_{n-1} * p|_{[t_{n-1},t_n]}])$$

We want to show that φ is a well-defined homomorphism; uniqueness follows.

To show well-definedness, we must demonstrate that φ is independent of our choice of (1) t_i, (2) j, (3) p up to homotopy (which is to say that for $p \sim q$, then $\varphi([p]) = \varphi([q])$), and (4) γ_i.

We will first show (2); that is, given

$$j' : \{0, \ldots, n-1\} \to \{1, 2\}$$

satisfying $p|_{[t_i,t_{i+1}]} \subseteq U_{j(i)}$, and making the abbreviation $q_i = \bar{\gamma}_i * p|_{[t_i,t_{i+1}]} * \gamma_{i+1}$, we want

$$\varphi_{j(i)}([q_i]) = \varphi_{j'(i)}([q_i])$$

If $j(i) \neq j'(i)$, then q is a loop in the intersection $U_1 \cap U_2$, and the agreement there of φ_1 and φ_2 yields the desired equality.

Next we show (4). We have

$$\varphi([p]) = *_{i=0}^{n-1} \varphi_{j(i)}([\bar{\gamma}_i * p|_{[t_i,t_{i+1}]} * \gamma_{i+1}])$$

Suppose that δ_i is another path from $p(t_i)$ to x. Then

$$\begin{aligned}
&\varphi_{j(i-1)}([\bar{\gamma}_{i-1} * p|_{[t_{i-1},t_i]} * \delta_i]) * \varphi_{j(i)}([\bar{\delta}_i * p|_{[t_i,t_{i+1}]} * \gamma_{i+1}]) \\
&\quad = \varphi_{j(i-1)}([\bar{\gamma}_{i-1} * p|_{[t_{i-1},t_i]} * \gamma_i]) * \varphi_{j(i-1)}([\bar{\gamma}_i * \delta_i]) \\
&\qquad * \varphi_{j(i)}([\bar{\delta}_i * \gamma_i]) * \varphi_{j(i)}([\bar{\gamma}_i * p|_{[t_i,t_{i+1}]} * \gamma_{i+1}]) \\
&\quad = \varphi_{j(i-1)}([\bar{\gamma}_{i-1} * p|_{[t_{i-1},t_i]} * \gamma_i]) \\
&\qquad * \varphi_{j(i)}([\bar{\gamma}_i * p|_{[t_i,t_{i+1}]} * \gamma_{i+1}])
\end{aligned}$$

since $\bar{\gamma}_i * \delta_i$ and $\bar{\delta}_i * \gamma_i$ are both paths in the intersection $U_1 \cap U_2$, and hence $\varphi_{j(i-1)}([\bar{\gamma}_i * \delta_i]) * \varphi_{j(i)}([\bar{\delta}_i * \gamma_i])$ cancels since φ_1 and φ_2 are consistent on $\pi_1(U_1 \cap U_2, x)$ and because $[\bar{\gamma}_i * \delta_i]^{-1} = [\bar{\delta}_i * \gamma_i]$. This proves (4).

Next, we show (1), that $\varphi([p])$ is independent of the choice of subdivision $t_0 < t_1 < \cdots < t_n$. Consider another subdivision $0 = s_0 < s_1 < \cdots < s_m = 1$. We want to show that

$$\varphi_{\mathbf{t}}([p]) = \varphi_{\mathbf{s}}([p])$$

We do this by demonstrating that both coincide with $\varphi_{\mathbf{r}}([p])$, where $\mathbf{r}$ is given by $\{r_i\} = \{t_i\} \cup \{s_i\}$.

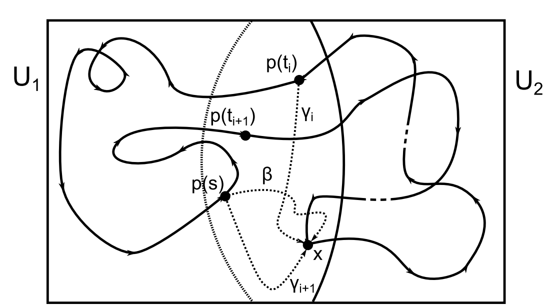
Thus, WLOG, we may assume that $\{t_i\} \subseteq \{s_i\}$. By induction on $m - n$, we can assume that $\{s_i\}$ is obtained by adding one element to $\{t_i\}$. Assume

$$0 < t_1 < \cdots < t_i < s < t_{i+1} < \cdots < 1$$

We want to show that

$$\begin{aligned}
&\varphi_{j(i)}([\bar{\gamma}_i * p|_{[t_i,t_{i+1}]} * \gamma_{i+1}]) \\
&\quad = \varphi_{j(i)}([\bar{\gamma}_i * p|_{[t_i,s]} * \beta]) * \varphi_{j(i)}([\bar{\beta} * p|_{[s,t_{i+1}]} * \gamma_{i+1}])
\end{aligned}$$

where β is defined, and we know that $j(i)$ is consistent, via the following diagram:



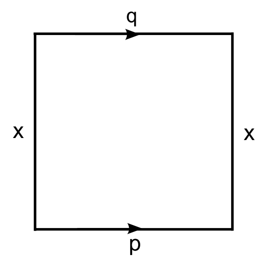
Since $\varphi_{j(i)}$ is a group homomorphism, (1) follows.

Next we will show that φ is indeed a group homomorphism. Say $p = p' * p''$; we want $\varphi([p]) = \varphi([p']) * \varphi([p''])$. Choose subdivisions $\{t_i\}_{i=0}^{n}$ and $\{s_j\}_{j=0}^{m}$ for $\varphi([p'])$ and $\varphi([p''])$ respectively. Condense these subdivisions together into a subdivision $\{r_k\}_{k=0}^{n+m+1}$ given by

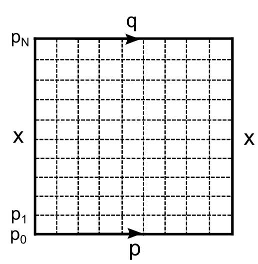
$$r_k = \begin{cases} \frac{t_k}{2} & k \leq n \\ \frac{s_k}{2} + \frac{1}{2} & k > n \end{cases}$$

Note that $p(r_k) \in U_1 \cap U_2$ and that $p|_{[r_k, r_{k+1}]} \subseteq U_l$ for $l \in \{1, 2\}$. Choose γ_m to be the constant path from $p(1/2) = x$ to x. This construction makes equality clear, and we omit the details (of which there are few).

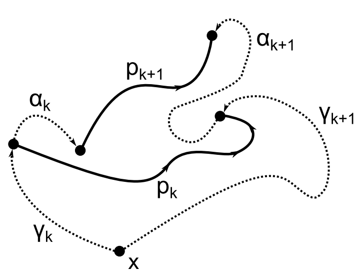
Finally, we show (2), that if $p \sim q$, then we also get $\varphi([p]) = \varphi([q])$. If $p \sim q$, then there is a homotopy $h : [0,1] \times [0,1] \to X$ such that $h(0,t) = p(t)$, $h(1,t) = q(t)$, $h(s,0) = h(s,1) = x$.



We know that $h^{-1}(U_1), h^{-1}(U_2)$ are an open cover of $[0,1] \times [0,1]$, so by the Lebesgue number lemma (see Munkres 27.5), $\exists \epsilon > 0$ such that $\forall y \in [0,1] \times [0,1]$, $B_\epsilon(y)$ is contained in either $h^{-1}(U_1)$ or $h^{-1}(U_2)$. Then there is $N \gg 0$ such that each square in the following grid represents a homotopy either in U_1 or in U_2:



We want to show that $\varphi([p_0]) = \varphi([p_N])$, and hence it suffices to show that $\varphi([p_k]) = \varphi([p_{k+1}])$. We present a picture, the details of which are left as an exercise.



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Lecture 36 — 11/29/10

Recall the Seifert-van Kampen Theorem: If X is a topological space of the form $X = U \cup V$ for $U, V \mathring{\subseteq} X$ where $U, V, U \cap V$ are all path-connected, then for $x \in U \cap V$, we have

$$\pi_1(X, x) \simeq \pi_1(U, x) \underset{\pi_1(U \cap V, x)}{*} \pi_1(V, x)$$

which is to say there is a pushout diagram

$$\begin{tikzcd} \pi_1(U \cap V, x) \arrow[r] \arrow[d] & \pi_1(V, x) \arrow[d] \\ \pi_1(U, x) \arrow[r] & \pi_1(X, x) \end{tikzcd}$$

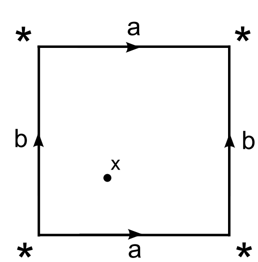
We will now provide some applications of this theorem, as well as some general applications of our study of algebraic topology, and we note that some details will be ommitted in our discussion.

Example.
1. $\pi_1(S^2, *) = 0$.
2. $\pi_1(\mathbb{C} - \{0, 1\}) \simeq \mathbb{Z} * \mathbb{Z}$.
3. The 2-torus of genus 1 is given by $T^2 = S^1 \times S^1$. We have $\pi_1(T^2) = \pi_1(S^1) \times \pi_1(S^1) \simeq \mathbb{Z} \times \mathbb{Z}$. We also know that T^2 has a universal cover $\mathbb{R}^2 \to T^2$ given by $(x, y) \mapsto (e^{2\pi i x}, e^{2\pi i y})$. So $T^2 \cong \mathbb{R}^2/(\mathbb{Z} \times \mathbb{Z})$.

Example. Let us determine the fundamental group of the punctured torus, $T^2 - \{*\}$. Our covering map above yields a covering

$$\mathbb{R}^2 - f^{-1}\{*\} \longrightarrow T^2 - \{*\}$$

Note that the preimage $f^{-1}\{*\}$ is a lattice in $\mathbb{R}^2$, and note also that $\mathbb{R}^2$ with this lattice removed is no longer simply connected. The punctured torus can be represented as a unit square in this lattice, given by



which represents a quotient wherein we identify the edges labeled a with one another, and separately the edges labeled b with one another With the point $*$ intact, this is the <u>fundamental polygon</u> of the torus.

Choose a base point x; we want to compute the group $\pi_1(T^2 - \{*\}, x)$. Let $U = T^2 - a$ and $V = T^2 - b$.

Then $U \cup V = T^2 - \{*\}$. We have $U \cong S^1 \times (0,1)$ and $V \cong (0,1) \times S^1$, and hence $U \cap V = (0,1) \times (0,1)$; note that all these spaces are path-connected. Since $\pi_1(U \cap V, x) = 0$, van-Kampen's theorem yields

$$\pi_1(T^2 - \{*\}, x) \simeq \mathbb{Z} * \mathbb{Z}$$

the free group on two generators. Note that

$$\pi_1(T^2 - \{*\}) \simeq \pi_1(\mathbb{C} - \{0,1\})$$

The space $\mathbb{C} - \{0,1\}$ is homeomorphic to the triply-punctured 3-sphere. The one-point compactification of $T^2 - \{*\}$ is T^2, but the one-point compactification of $\mathbb{C} - \{0,1\}$ is S^3 with three points identified. That is, though we have an isomorphism of fundamental groups, we still find that

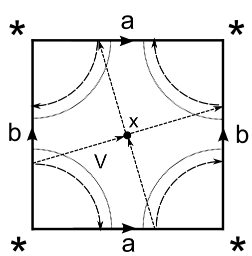
$$T^2 - \{*\} \not\cong \mathbb{C} - \{0,1\}$$

Note lastly that the group $\pi(T^2 - \{*\}, x)$ is generated by $[\alpha], [\beta]$ where α and β are the lines of latitude and longitude, respectively, which intersect x.

Now take $U = T^2 - \{*\}$ and let V be a small open disc about x. Then $T^2 = U \cup V$, and we will compute $\pi_1(T^2, x)$. We know that $\pi_1(U, x) \simeq F(\{[\alpha], [\beta]\}) \simeq \mathbb{Z} * \mathbb{Z}$ and $\pi_1(V, x) = 0$. Since $U \cap V$ is the punctured open disc, $\pi_1(U \cap V, x) \simeq \mathbb{Z}$. By van-Kampen's theorem,

$$\pi_1(T^2, x) \simeq (\mathbb{Z} * \mathbb{Z}) *_{\mathbb{Z}} 0$$

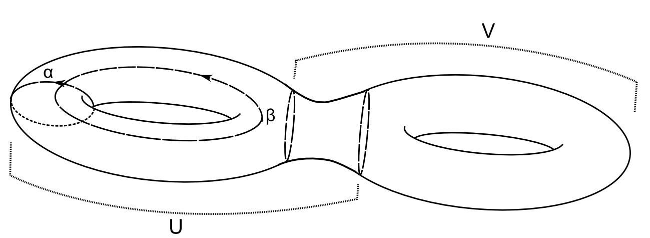
This is the free group on $[\alpha], [\beta]$ quotiented by the normal subgroup generated by the image of the homomorphism $\mathbb{Z} \to \mathbb{Z} * \mathbb{Z}$ induced by inclusion. We can deform a generator of $\mathbb{Z}$ (a circular path) given this inclusion



So $\pi_1(T^2, x)$ is the quotient of $\mathbb{Z} * \mathbb{Z}$ by the normal subgroup generated by the commutator

$$[[\alpha], [\beta]] = [\alpha][\beta][\alpha^{-1}][\beta^{-1}]$$

Example. Now consider the two-holed torus Σ given by



Both U and V are the torus T^2 with a disc removed; hence, $\pi_1(U) \simeq \pi_1(V) \simeq \mathbb{Z} * \mathbb{Z}$. We also have $U \cap V \cong S^1 \times (0,1)$, so $\pi_1(U \cap V) \simeq \mathbb{Z}$. Then van-Kampen's theorem gives

$$\pi_1(\Sigma) \simeq (\mathbb{Z} * \mathbb{Z}) *_{\mathbb{Z}} (\mathbb{Z} * \mathbb{Z})$$

This is the free group generated by $[\alpha], [\beta], [\alpha'], [\beta']$ with a single relation, $[[\alpha], [\beta]] = [[\alpha'], [\beta']]^{-1}$, or equivalently, $[[\alpha], [\beta]][[\alpha'], [\beta']]$.

In general, the torus of genus g (with g holes), Σ_g, has fundamental group $\pi_1(\Sigma_g)$ given by the free group on $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ modulo $[\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g]$.

Definition 36.1. Let $f, g : X \to Y$ be continuous maps between topological spaces. A homotopy from f to g is a map $h : X \times [0,1] \to Y$ such that $h|_{X \times \{0\}} = f$ and $h|_{X \times \{1\}} = g$.

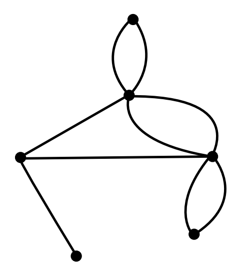
Definition 36.2. A topological space X is contractible if $\mathrm{id} : X \to X$ is homotopic to some constant map.

Example. $\mathbb{R}^n$ is contractible by the contraction

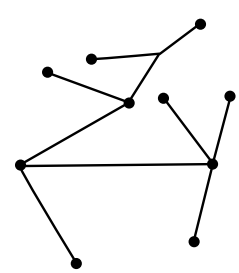
$$\begin{aligned} \mathbb{R}^n \times [0,1] &\longrightarrow \mathbb{R}^n \\ (v, t) &\longmapsto tv \end{aligned}$$

Note. If X is contractible, then $\pi_1(X)$ is always trivial.

Definition 36.3. A graph is a space akin to



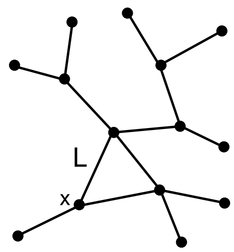
A graph is a tree if it has no loops



Claim 36.4. *Any tree is contractible and hence has trivial fundamental group.*

Note. Any connected graph can be made into a tree by deleting midpoints of edges. Moreover, any connected graph can be made simply-connected by deleting midpoints of *finitely many* edges.

Observation 36.5. Suppose our graph G only requires one deletion.



We claim that there is a projection $p : G \to L$ that is homotopic to id_G. So $\pi_1(G, x) \simeq \pi_1(L, x) \simeq \mathbb{Z}$.

Observation 36.6. Suppose G becomes simply connected after deleting finitely many points $x_1, \ldots, x_d$. Let

$$U_i = G - \{x_1, \ldots, \hat{x_i}, \ldots, x_d\}$$

We have $\pi_1(U_i) \simeq \mathbb{Z}$ by our previous discussion. Moreover, $G = \bigcup U_i$. Then van-Kampen's theorem yields

$$\pi_1(G) \simeq \pi_1(U_1) * \cdots * \pi_1(U_d)$$

the free group on d generators. In general, if G is a connected graph, then $\pi_1(G)$ is free; in the infinite case, we write G as the infinite union of finite graphs.